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Theorems on the Attraction of Ellipsoids for Certain Laws of Force Other than the Inverse Square

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XXIII. *Theorems on the Attraction of Ellipsoids for certain Laws of Force other than the Inverse Square.*

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Contents.

1. To find the potential at an internal point P of a thin ellipsoidal shell bounded by similar concentric ellipsoids, the law of force being the inverse κth power of the distance.

Let (x, y, z) be the coordinates of P; (l, m, n) the direction cosines of a radius vector drawn from P to any point Q on the surface; and let PQ $= r$. Let (a, b, c) be the semi-axes of the ellipsoids, and (α, β, γ) their squared reciprocals. Then from the equation of the ellipsoid

$$(\alpha l^2 + \beta m^2 + \gamma n^2) r^2 + 2(\alpha l x + \beta m y + \gamma n z) r - (1 - \alpha x^2 - \beta y^2 - \gamma z^2) = 0 \quad . \quad (1).$$

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For the sake of brevity we shall put

$$\left.\begin{aligned} 1/\mathrm{D}^2 &= \alpha l^2 + \beta m^2 + \gamma n^2 \\ \mathrm{F} &= \alpha lx + \beta my + \gamma nz \\ \mathrm{E} &= 1 - \alpha x^2 - \beta y^2 - \gamma z^2 \end{aligned}\right\} \quad \ldots \ldots \quad (2).$$

Thus D is the semi-diameter parallel to the radius vector r, F is a linear function of (x, y, z) such that $\mathrm{F} = 0$ represents a plane through the centre O diametral to PQ, and $\mathrm{E} = 0$ is the equation of the ellipsoid.

The quadratic may therefore be written in either of the forms

$$\left.\begin{aligned} r^2 + 2\mathrm{FD}^2 r - \mathrm{ED}^2 &= 0 \\ u^2 - \frac{2\mathrm{F}}{\mathrm{E}} u - \frac{1}{\mathrm{ED}^2} &= 0 \end{aligned}\right\} \quad \ldots \ldots \quad (3)$$

where u is the reciprocal of r. The roots r_1, r_2 have opposite signs and

$$r_1 + r_2 = -2\mathrm{FD}^2, \qquad r_1 r_2 = -\mathrm{ED}^2 \quad \ldots \ldots \quad (4).$$

Taking together the two elements of the shell enclosed by an elementary cone whose vertex is the attracted point P and whose solid angle is $d\omega$, the potential of the whole shell is evidently

$$\mathrm{V} = \frac{1}{\kappa - 1} \iint \left\{ \frac{\rho_1 r_1^2 \, d\omega \, dr_1}{r_1^{\kappa - 1}} + \frac{\rho_2 r_2^2 \, d\omega \, (-dr_2)}{(-r_2)^{\kappa - 1}} \right\} \quad \ldots \ldots \quad (5),$$

where r_1, r_2 have their proper signs as given by the standard quadratic (3), r_1 being the positive root and r_2 the negative root. Also ρ_1, ρ_2 are the densities of the shell at the extremities of the chord through P.

It is evident that if we integrate all round the point P every element of the surface is taken twice over. *We must therefore halve the result given by* (5).

2. The values of dr_1, dr_2 depend on the kind of shell under consideration. Supposing that the shell is a thin homœoid, we differentiate the quadratic (1) on the supposition that b/a and c/a are constant. We thus find

$$dr = \frac{da}{a} \cdot \frac{\mathrm{D}^2}{r + \mathrm{FD}^2}.$$

Using the relations (2) and (4), it immediately follows that

$$dr_1 = \frac{da}{a} \frac{2\mathrm{D}^2}{r_1 - r_2}, \qquad dr_2 = \frac{da}{a} \frac{2\mathrm{D}^2}{r_2 - r_1} \quad \ldots \ldots \quad (6).$$

These values of dr_1, dr_2 may easily be verified by a short geometrical proof.

The potential at an internal point, P, of a thin homœoid is therefore

$$V = \frac{1}{\kappa - 1} \frac{da}{a} \iint \frac{\rho_1 r_1^{3-\kappa} + \rho_2 (-r_2)^{3-\kappa}}{r_1 - r_2} D^2 \, d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

where the integration extends all round the point P. Writing $da/a = \mu$, this formula also represents the potential of a thin stratum placed on the ellipsoid whose surface-density at any point Q is $\mu\rho p$, where p is the perpendicular from the centre on the tangent plane at Q.

3. *If the law of force is any even power of the distance,* κ is even, *and if the density is uniform and equal to unity the expression* (7) takes the form

$$V = \frac{1}{\kappa - 1} \frac{da}{a} \iint \frac{r_1^{3-\kappa} - r_2^{3-\kappa}}{r_1 - r_2} D^2 \, d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8),$$

in which the subject of integration is a symmetrical function of the roots of the fundamental quadratic (3). Since such a function can be expressed as an integral rational function of the roots, the integration is much facilitated.

If κ be greater than 3, it will be more convenient to write $r = 1/u$. Supposing κ to be an even integer the formula (7) then becomes

$$V = \frac{1}{\kappa - 1} \frac{da}{a} \frac{1}{E} \iint \frac{\rho_1 u_1^{\kappa-3} - \rho_2 u_2^{\kappa-3}}{u_1 - u_2} \, d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

If the law of force is any odd power of the distance and the density is uniform, the potential at an internal point becomes

$$V = \frac{1}{\kappa - 1} \frac{da}{a} \iint \frac{r_1^{3-\kappa} + r_2^{3-\kappa}}{r_1 - r_2} D^2 \, d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10).$$

This is not a symmetrical function of the roots of the quadratic. It appears, therefore, that the odd powers of the force lead to a more complicated analysis than the even ones.

4. *We may apply a similar method to solid ellipsoids.* Not to complicate matters at the beginning of the investigation, let us suppose that the ellipsoid is homogeneous and of unit density. The potential of the elementary cone at its vertex is found by integrating from $r = 0$ to r. The potential of the whole solid at an internal point P is therefore

$$V = \tfrac{1}{2} \frac{-1}{(\kappa - 1)(\kappa - 4)} \iint (r_1^{4-\kappa} + (-r_2)^{4-\kappa}) \, d\omega + C \quad . \quad . \quad . \quad . \quad (11),$$

where the result has been halved for the same reason as before. Writing $d\omega = \sin\theta \, d\theta \, d\phi$ the integration extends from $\phi = 0$ to 2π, $\theta = 0$ to π.

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When κ is negative or positive and less than 4, the constant C is zero. When κ is positive and greater than 4 the constant is infinite. Since the potential of a small sphere of fixed radius ϵ and unit density at its centre is an absolute constant, we may take the lower limit in the integration for r to be ϵ, and omit the potential of the small sphere. The value of C will then be finite and depend on the magnitude of ϵ. As it will disappear when V is differentiated to find the forces, we shall omit it here also.

5. Before discussing the general values of the expressions (7) and (8), it will be useful to notice a few particular cases.

Suppose that the ellipsoidal shell is homogeneous and that the force varies as the inverse fourth power of the distance. We then have $\kappa = 4$, and the potential at any internal point P is

$$V = \frac{\mu}{3}\iint \frac{-D^2}{r_1 r_2}\, d\omega = \frac{4\pi\mu}{3}\frac{1}{E}$$

by using the standard relations (4). Here, as before,

$$E = 1 - \alpha x^2 - \beta y^2 - \gamma z^2.$$

Thus *the level surfaces inside the homœoid are quadrics similar and similarly situated to the bounding surface.* This agrees with the result given by TOWNSEND, in the 'Quarterly Journal,' vol. 12, 1873, "on the attraction of the ellipsoid for the law of inverse fourth power of the distance."

The potential V′ *at an external point* $(x'y'z')$ follows immediately from that at an internal point. We have

$$V' = \frac{4\pi\mu}{3}\,\frac{abc}{a'b'c'}\cdot\frac{1}{E'}$$

where E′ may be written on either of the forms

$$E' = 1 - \frac{a^2x'^2}{a'^4} - \frac{b^2y'^2}{b'^4} - \frac{c^2z'^2}{c'^4} = \epsilon^2 \cdot \left(\frac{x'^2}{a'^4} + \frac{y'^2}{b'^4} + \frac{z'^2}{c'^4}\right)$$

and

$$\frac{x'^2}{a^2+\epsilon^2} + \frac{y'^2}{b^2+\epsilon^2} + \frac{z'^2}{c^2+\epsilon^2} = 1.$$

Here a', b', c' are the semi-axes of the confocal through the attracted point.

The potential of a solid ellipsoid at an internal point may also be easily found for the same law of force. The integration for r, in equation (11), takes a logarithmic form when $\kappa = 4$, and we have

$$V = \tfrac{1}{2}\,\tfrac{1}{3}\int d\omega\,\{\log r_1 + \log(-r_2)\} = \tfrac{1}{6}\int d\omega \log(ED^2)$$

by using the standard forms (4). Integrating and omitting the constant, viz.: $\int d\omega \log D^2$, we have

$$V = \frac{2\pi}{3} \log E.$$

From this we easily find, by differentiating with regard to x, that the force at *an internal point* resolved parallel to x is

$$X = -\frac{4\pi}{3} \frac{1}{E} \frac{x}{a^2}.$$

By IVORY'S theorem it immediately follows that the force at *an external point* (x', y', z') is

$$X' = -\frac{4\pi}{3} \frac{abc}{a'b'c'} \cdot \frac{x'}{a'^2} \frac{1}{\epsilon^2} \left(\frac{x'^2}{a'^4} + \frac{y'^2}{b'^4} + \frac{z'^2}{c'^4}\right)^{-1}.$$

These values of X, X′ agree with those given by TOWNSEND in the memoir already referred to.

We may deduce from the values of X′, Y′, Z′ the potential at any external point. Since

$$\frac{x'^2}{a^2 + \epsilon^2} + \frac{y'^2}{b^2 + \epsilon^2} + \frac{z'^2}{c^2 + \epsilon^2} = 1,$$

we have

$$\frac{2x'}{a^2 + \epsilon^2} = \left\{\frac{x'^2}{(a^2 + \epsilon^2)^2} + \frac{y'^2}{(b^2 + \epsilon^2)^2} + \frac{z'^2}{(c^2 + \epsilon^2)^2}\right\} \frac{d\epsilon^2}{dx'}.$$

Hence

$$X' = -\frac{2\pi}{3} \frac{abc}{a'b'c'} \frac{1}{\epsilon^2} \frac{d\epsilon^2}{dx'}.$$

It follows by an easy integration that *the potential of a homogeneous solid ellipsoid at an external point for the law of the inverse fourth is*

$$V' = -\frac{2\pi}{3} \int_{\infty}^{\epsilon^2} \frac{abc}{a'b'c'} \frac{d\lambda}{\lambda}.$$

We notice that the potential is constant if ϵ^2 is constant; thus the *external level surfaces are confocals.*

Let p' be the perpendicular from the centre on a tangent plane to a confocal, then $2p'\,dp' = d\lambda$. The force normal to the confocal is therefore

$$F = \frac{dV'}{dp'} = \frac{dV'}{d\lambda} \frac{d\lambda}{dp'} = -\frac{4\pi}{3} \frac{abc}{a'b'c'} \frac{p'}{\epsilon^2},$$

where a', b', c' are the semi-axes of the confocal and $\epsilon^2 = a'^2 - a^2$. *For points on the* *same confocal the normal force is therefore proportional to the perpendicular on the tangent plane.*

The law of the inverse fourth power derives some importance from a theorem of JELLETT'S. JELLETT showed that, if V_κ be the potential when the law of force is the inverse κ^{th} power of the distance,

$$V_{\kappa+2} = \frac{1}{(\kappa+1)(\kappa-2)}\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}+\frac{d^2}{dz^2}\right)V_\kappa.$$

By using this theorem, we may deduce the potential for any inverse integral power greater than the square, when the potential for the two laws of the inverse distance and the inverse fourth power are completely known. This theorem was communicated to the British Association at its meeting in Dublin in 1857, and briefly alluded to, rather than explained, in the report at page 3. But TOWNSEND, in the 'Quarterly Journal,' vol. 16, 1879, has called attention to the theorem and used it to find the potentials of circular rings and plates, &c., for various laws of force.

In this way the potential, when the force varies as the inverse sixth power, may be derived from the potential for the inverse fourth, but as some of these potentials have the function we have called E in the denominator, while others have the still more complicated function $a'b'c'E'$ in the same position, the labour will be found to become very great as we proceed to higher laws of force.

We notice that the potentials of a homogeneous thin homœoid, at both external and internal points, have been expressed as algebraic functions of the coordinates of the attracted point. It follows from JELLETT'S theorem that these potentials for all even higher laws of force can also be expressed free from all signs of integration. A similar argument will obviously apply to several other cases.

TOWNSEND also observes that JELLETT'S theorem remains true when the number of variables is extended beyond three. Thus, if

$$V_\kappa = \Sigma\,\frac{m}{r^\kappa}, \quad r^2 = (x_1-a_1)^2+(x_2-a_2)^2+\ldots+(x_n-a_n)^2$$

we find by simple differentiation that

$$V_{\kappa+2} = \frac{1}{(\kappa+1)(\kappa+1-n)}\left(\frac{d^2}{dx_1^{\,2}}+\ldots+\frac{d^2}{dx_n^{\,2}}\right)V_\kappa.$$

6. *Suppose the ellipsoidal shell to be homogeneous and the force to vary as the inverse sixth power of the distance.* We then have $\kappa = 6$, and the potential at any internal point P is by equation (8)

$$V = \frac{\mu}{5}\iint (r_1^{\,2}+r_1r_2+r_2^{\,2})\,\frac{D^2\,d\omega}{-\,r_1^{\,3}r_2^{\,3}}\,.$$

Using the formulæ (4) this becomes

$$V = \frac{\mu}{5}\iint(4F^2D^4 + ED^2)\frac{d\omega}{D^4E^3}$$
$$= \frac{\mu}{5}\left[\frac{4}{E^3}\iint(\alpha lx + \beta my + \gamma nz)^2\,d\omega + \frac{1}{E^2}\iint(\alpha l^2 + \beta m^2 + \gamma n^2)\,d\omega\right].$$

Writing $d\omega = \sin\theta\,d\theta\,d\phi$, as usual, and putting $\cos^2\theta$ for l^2, m^2, n^2 in turn, these integrals follow at once; the terms with the products lm, mn, nl being obviously zero. We therefore have

$$V = \frac{\mu}{5}\frac{4\pi}{3}\left[\frac{4(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}{(1 - \alpha x^2 - \beta y^2 - \gamma z^2)^3} + \frac{\alpha + \beta + \gamma}{(1 - \alpha x^2 - \beta y^2 - \gamma z^2)^2}\right].$$

The potential at an external point x', y', z' is therefore

$$V' = \frac{4\pi\mu}{15}\frac{abc}{a'b'c'}\left(\frac{p'}{\epsilon}\right)^4\left[\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} + \frac{4}{\epsilon^2} - 4p'^2\left(\frac{x'^2}{a'^6} + \frac{y'^2}{b'^6} + \frac{z'^2}{c'^6}\right)\right],$$

where a', b', c' are the semi-axes of the confocal through the attracted point, $\epsilon^2 = a'^2 - a^2$, and p' is the perpendicular on the tangent plane at the attracted point.

The potential of a solid homogeneous ellipsoid at an internal point for the inverse sixth power is easily seen to be

$$V = -\frac{2\pi}{15}\left\{2\cdot\frac{\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2}{E^2} + \frac{\alpha + \beta + \gamma}{E}\right\}.$$

The potential at an external point is given later.

7. Another interesting case is when the index κ of the law of force is zero or negative.

Taking $\kappa = 0$ so that the force is independent of the distance, we see from equation (8) that the potential of a homogeneous thin homœoid at an internal point is

$$V = \frac{\mu}{\kappa - 1}\iint(r_1^2 + r_1r_2 + r_2^2)\,D^2\,d\omega.$$

Substituting (exactly as in the last article) from the standard forms (4) we see at once that the potential takes the form

$$V = Ax^2 + By^2 + Cz^2 + D$$

where A, B, C, D are constants which depend on the axes of the ellipsoid.

In the same way when $\kappa = -2$, or the force varies as the square of the distance, we find that V is a quartic function of the coordinates of the attracted point containing only even powers.

8. *To find the potential of a thin homogeneous homœoid at an internal point when the force varies as any even inverse power of the distance greater than the square.*

Putting $u = 1/r$ and using the formulæ (4), the potential (8) becomes

$$V = \frac{\mu}{\kappa - 1} \frac{1}{E} \iint \frac{u_1^{\kappa-3} - u_2^{\kappa-3}}{u_1 - u_2} d\omega,$$

where u_1, u_2 are the roots of the quadratic

$$u^2 - \frac{2F}{E} u - \frac{1}{ED^2} = 0.$$

Writing the quadratic in the form $u^2 - pu - q = 0$, we have by a known theorem

$$u_1^n + u_2^n = p^n + np^{n-2} q + \ldots + n \frac{L(n-f-1)}{L(f) L(n-2f)} p^{n-2f} q^f + \ldots,$$

where $L(f) = 1 . 2 . 3 \ldots f$. This series stops at the first negative power of p, so that the number of terms is $\frac{1}{2}n + 1$ or $\frac{1}{2}(n+1)$ according as n is even or odd.

We find by simply differentiating that

$$\frac{u_1^n - u_2^n}{u_1 - u_2} = \frac{1}{n} \frac{d}{dp} (u_1^n + u_2^n),^*$$

the differentiation being performed on the supposition that $u_1 u_2 = -q$ is constant.

We thus find that

$$\frac{u_1^n - u_2^n}{u_1 - u_2} = p^{n-1} + (n-2) p^{n-3} q + \ldots + \frac{L(n-f-1}{L(f) . L(n-2f-1)} p^{n-2f-1} q^f + \ldots$$

In order to find V, we require the value of the integral

$$\iint p^{\kappa-4-2f} q^f d\omega = \iint \left(\frac{2F}{E}\right)^{\kappa-4-2f} \left(\frac{1}{ED^2}\right)^f d\omega.$$

* One case of this equation is given by SERRET, who uses it to find the expansion of $x^n - 1/x^n$ in powers of $x + 1/x$, when that of $x^n + 1/x^n$ is known. Probably the equation for this case was suggested by differentiating $\cos n\theta$. See 'Cours d'Algèbre Supérieure,' fourteenth lesson.

We know that

$$\iint (\lambda l + \mu m + \nu n)^{2s}\, d\omega = (\lambda^2 + \mu^2 + \nu^2)^s \frac{4\pi}{2s+1}.$$

Hence in our case

$$\iint (\alpha l x + \beta m y + \gamma n z)^{2s}\, d\omega = (\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)^s \frac{4\pi}{2s+1}.$$

Operating with

$$\nabla = \frac{1}{\alpha}\frac{d^2}{dx^2} + \frac{1}{\beta}\frac{d^2}{dy^2} + \frac{1}{\gamma}\frac{d^2}{dz^2}$$

on both sides, we have

$$\iint (\alpha l x + \beta m y + \gamma n z)^{2s-2}\,(\alpha l^2 + \beta m^2 + \gamma n^2)\, d\omega = \frac{4\pi}{(2s+1)\,2s\,(2s-1)}\nabla \mathrm{P}.$$

Repeating this process, and writing $2s = \kappa - 4$,

$$\iint \mathrm{F}^{\kappa-4-2f}\left(\frac{1}{\mathrm{D}^2}\right)^f d\omega = \frac{4\pi\,\mathrm{L}(\kappa-4-2f)}{\mathrm{L}(\kappa-3)}\nabla^f \mathrm{P},$$

where

$$\mathrm{P} = (\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)^{\frac{1}{2}(\kappa-4)}.$$

The potential of the thin homœoid when the force varies as the inverse κth power of the distance is therefore

$$\mathrm{V} = \frac{2\pi\mu}{(\kappa-1)(\kappa-3)}\left(\frac{2}{\mathrm{E}}\right)^{\kappa-3}\left\{1 + \frac{1}{2^2}\frac{\mathrm{E}}{\kappa-4} + \frac{1}{2^4}\frac{\mathrm{E}^2\nabla^2}{1\,.\,2\,.\,(\kappa-4)(\kappa-5)} + \ldots\right\}\mathrm{P}.$$

The general term within the bracket is

$$\frac{1}{2^{2f}}\frac{\mathrm{L}(\kappa-4-f)}{\mathrm{L}(f)\,.\,\mathrm{L}(\kappa-4)}\mathrm{E}^f\nabla^f,$$

and ∇, P have the meanings given above, viz.,

$$\mathrm{E} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2},$$

$$\mathrm{P} = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}(\kappa-4)},$$

$$\nabla = a^2\frac{d^2}{dx^2} + b^2\frac{d^2}{dy^2} + c^2\frac{d^2}{dz^2}.$$

It follows, therefore, that the potential of the homœoid has been found free from all sign of integration. It has the form

$$\mathrm{V} = \mathrm{A}\frac{\mathrm{P}}{\mathrm{E}^{\kappa-3}} + \mathrm{B}\frac{\nabla\mathrm{P}}{\mathrm{E}^{\kappa-4}} + \mathrm{C}\frac{\nabla^2\mathrm{P}}{\mathrm{E}^{\kappa-5}} + \ldots$$

where A, B, C are numerical factors whose values have just been found. The numerators of the powers of E are homogeneous functions of x^2, y^2, z^2, whose dimensions are continually reduced by the differentiations implied in the operator ∇.

The series has $\frac{1}{2}(\kappa - 2)$ terms. Thus for the law of the inverse fourth it reduces to the one term given in Art. 5; for the law of the inverse sixth power it becomes the two terms found in Art. 6, and so on.

The potential at an external point P′, whose coordinates are (x', y', z'), follows from that at an internal point by what is practically a method of inversion. We describe through P′ a confocal homœoid whose axes are a', b', c', and of the same volume as the original homœoid. The potentials of the two homœoids at corresponding points are then equal. Remembering the transformation already made in Art. 5, we find that the potential at an external point is

$$V' = \frac{2\pi\mu}{(\kappa-1)(\kappa-3)} \frac{abc}{a'b'c'} \left(\frac{2}{E'}\right)^{\kappa-3} \left\{1 + \frac{1}{2^2} \frac{E'\nabla'}{\kappa-4} + \frac{1}{2^4} \frac{E'^2\nabla'^2}{1.2(\kappa-4)(\kappa-5)} + \ldots\right\} P',$$

where E′ and P′ have either of the forms

$$E' = 1 - \frac{a^2x'^2}{a'^4} - \frac{b^2y'^2}{b'^4} - \frac{c^2z'^2}{c'^4} = \epsilon^2\left(\frac{x'^2}{a'^4} + \frac{y'^2}{b'^4} + \frac{z'^2}{c'^4}\right),$$

$$P' = \left(\frac{a^2x'^2}{a'^6} + \frac{b^2y'^2}{b'^6} + \frac{c^2z'^2}{c'^6}\right)^{\frac{1}{2}(\kappa-4)},$$

$$\nabla' = \frac{a'^4}{a^2}\frac{d^2}{dx'^2} + \frac{b'^4}{b^2}\frac{d^2}{dy'^2} + \frac{c'^4}{c^2}\frac{d^2}{dz'^2}.$$

It should be noticed that the differentiations implied in the operator ∇' are to be performed on (x', y', z') on the supposition that a', b', c' are constant.

9. *To find the potential of a thin homogeneous homœoid at an internal point when the force varies as the inverse κth power of the distance and* $\kappa = 2$, *zero, or any even negative number.*

We now use the potential (8) in the form

$$V = \frac{\mu}{\kappa - 1}\iint \frac{r_1^{3-\kappa} - r_2^{3-\kappa}}{r_1 - r_2} D^2\, d\omega$$

where r_1, r_2 are the roots of the quadratic

$$r^2 + 2FD^2r - ED^2 = 0.$$

Putting $p = -2FD^2$, $q = ED^2$, we substitute in the series

$$\frac{r_1^{3-\kappa} - r_2^{3-\kappa}}{r_1 - r_2} = p^{2-\kappa} + (1-\kappa)\,p^{-\kappa}q + \frac{(-\kappa)(-\kappa-1)}{1.2}\,p^{-\kappa-2}q^2 + \ldots$$

This series has $\frac{1}{2}(4-\kappa)$ terms, and the general term has been written at length in Art. 8.

To determine the potential we must therefore find the value of

$$\iint p^{2-\kappa-2f} q^f \mathrm{D}^2 \, d\omega = \mathrm{E}^f \iint (2\mathrm{F})^{2-\kappa-2f} (\mathrm{D}^2)^{3-\kappa-f} \, d\omega$$

where f has any positive value from zero to $\frac{1}{2}(2-\kappa)$. To effect this integration we shall use the following lemma.

10. *Lemma.* By a known theorem we have

$$\iint \frac{d\omega}{\alpha l^2 + \beta m^2 + \gamma n^2} = 2\pi \int_0^\infty \frac{v^{-\frac{1}{2}} \, dv}{\mathrm{Q}},$$

where

$$\mathrm{Q}^2 = (\alpha + v)(\beta + v)(\gamma + v).$$

Now

$$\left(\frac{d}{d\alpha} + \frac{d}{d\beta} + \frac{d}{d\gamma}\right)\frac{1}{\mathrm{Q}} = \frac{d}{dv}\frac{1}{\mathrm{Q}}.$$

Hence, operating t times on the equation with the left-hand operator, we find

$$\iint \frac{d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{t+1}} = \frac{2\pi(-1)^t}{\mathrm{L}(t)} \int_0^\infty v^{-\frac{1}{2}} \, dv \left(\frac{d}{dv}\right)^t \frac{1}{\mathrm{Q}}.$$

Differentiating again f times with regard to α, g times with regard to β, h times with regard to γ, we have

$$\iint \frac{l^{2f} m^{2g} n^{2h} \, d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{s+t+1}} = \mathrm{K} \, . \int_0^\infty v^{-\frac{1}{2}} \, dv \left(\frac{d}{dv}\right)^t \frac{1}{(\alpha + v)^f (\beta + v)^g (\gamma + v)^h \mathrm{Q}},$$

where

$$\mathrm{K} = \frac{2\pi(-1)^t}{\mathrm{L}(s+t)} \cdot \frac{\mathrm{L}(2f)}{\mathrm{L}(f)} \cdot \frac{\mathrm{L}(2g)}{\mathrm{L}(g)} \frac{\mathrm{L}(2h)}{\mathrm{L}(h)} \frac{1}{2^{2s}},$$

where $s = f + g + h$ and $\mathrm{L}(f) = 1 \, . \, 2 \, . \, 3 \ldots f$.

By an expansion we deduce from this

$$\iint \frac{(\lambda l + \mu m + \nu n)^{2s} \, d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{s+t+1}} = \mathrm{K}' \int_0^\infty v^{-\frac{1}{2}} \, dv \left(\frac{d}{dv}\right)^t \left\{\frac{\lambda^2}{\alpha + v} + \frac{\mu^2}{\beta + v} + \frac{\nu^2}{\gamma + v}\right\}^s \frac{1}{\mathrm{Q}},$$

where

$$\mathrm{K}' = \frac{\mathrm{L}(2s)}{\mathrm{L}(s)\,\mathrm{L}(s+t)} \cdot \frac{2\pi(-1)^t}{2^{2s}}.$$

The expansion is not difficult, for the odd powers of l, m, n, on the left hand side give zero on integration, and the general integral for any set of even powers is written down just above.

The subject of integration on the right hand side is infinite when $v = 0$, but it is not difficult to see that the integral from zero to any very small value of v is ultimately zero.

11. To apply the lemma to the integral required in our proposition we refer to Art. 1 for the definitions of F and D^2. We have

$$F = \alpha lx + \beta my + \gamma nz.$$

We therefore put $\lambda = \alpha x$, $\mu = \beta y$, $\nu = \gamma z$, $2s = 2 - \kappa - 2f$, $s + t + 1 = 3 - \kappa - f$. This gives $t = \frac{1}{2}(2 - \kappa)$ so that t is a positive integer when κ is even and not greater than 2. We thus have

$$V = \frac{\mu}{\kappa - 1}\iint \Sigma \frac{L(2 - \kappa - f)}{L(f) . L(2 - \kappa - 2f)} . E^f . (2F)^{2-\kappa-2f}(D^2)^{3-\kappa-f} d\omega,$$

$$= \frac{2\pi\mu}{\kappa - 1}(-1)^t \int_0^\infty v^{-\frac{1}{2}} dv \left(\frac{d}{dv}\right)^t \Sigma \frac{E^f}{L(f)} \frac{1}{L(s)} \left\{\frac{\alpha^2 x^2}{\alpha + v} + \ldots\right\}^s \frac{1}{Q}.$$

The summation extends from $f = 0$ to $\frac{1}{2}(2 - \kappa)$ and s varies from $\frac{1}{2}(2 - \kappa)$ to 0. The Σ is therefore equivalent to an expansion by the binomial theorem. We thus find for the required potential

$$V = \frac{2\pi\mu}{\kappa - 1} \frac{(-1)^t}{L(t)} \int_0^\infty v^{-\frac{1}{2}} dv \left(\frac{d}{dv}\right)^t . \left\{E + \frac{\alpha^2 x^2}{\alpha + v} + \frac{\beta^2 y^2}{\beta + v} + \frac{\gamma^2 z^2}{\gamma + v}\right\}^t \frac{1}{Q}.$$

Substituting for E its value (given in Art. 1), the potential becomes

$$V = \frac{2\pi\mu(-1)^t}{(\kappa - 1) L(t)} \int_0^\infty v^{-\frac{1}{2}} dv \left(\frac{d}{dv}\right)^t \left\{1 - \frac{\alpha v x^2}{\alpha + v} - \frac{\beta v y^2}{\beta + v} - \frac{\gamma v z^2}{\gamma + v}\right\}^t \frac{1}{Q},$$

where $t = \frac{1}{2}(2 - \kappa)$.

Writing $v = 1/u$ we find that the potential of a thin homœoid when the index κ of the force is even and not greater than 2 at an internal point, is

$$V = \frac{2\pi\mu}{\kappa - 1} \int_0^\infty u^{-\frac{3}{2}} du \left(u^2 \frac{d}{du}\right)^t \frac{abc\, u^{\frac{3}{2}}}{Q_1} . \frac{R^t}{L(t)},$$

where

$$R = 1 - \frac{x^2}{a^2 + u} - \frac{y^2}{b^2 + u} \quad \frac{z^2}{c^2 + u}$$

$$Q_1^2 = (a^2 + u)(b^2 + u)(c^2 + u).$$

At an external point $(x'y'z')$ *the potential is*

$$V' = \frac{2\pi\mu}{\kappa - 1}\int_{\epsilon^2}^{\infty}(u - \epsilon^2)^{-\frac{3}{2}}\, du \left\{(u - \epsilon^2)^2 \frac{d}{du}\right\}^t \frac{abc\,(u - \epsilon^2)^{\frac{3}{2}}}{Q_1}\left(\frac{u}{u - \epsilon^2}\right)^t \frac{R'^t}{L\,(t)}$$

where R′ differs from R only in having $(x'y'z')$ written for xyz.

Here $t = \frac{1}{2}(2 - \kappa)$, so that when $t = 0$ the formulæ reduce to the well-known results for the law of the inverse square.

It is easy to verify these results in the case of a spherical shell.

12. *To find the potential at an internal point* P *of a thin heterogeneous homœoid when the law of force is an inverse even positive power of the distance, and the density at any point* (ξ, η, ζ), *is* $\phi(\xi, \eta, \zeta)$.

The coordinates of the attracted point P being (x, y, z), the coordinates of any point Q on the surface are $\xi = x + lr$, &c. If ρ_1, ρ_2 are the densities at the extremities of a chord through P, we have

$$\rho_1 = \phi\,(x + lr_1, y + mr_1, z + nr_1) = \phi + r_1\,\delta\phi + \frac{r_1^2}{1\,.\,2}\,\delta^2\phi + \ldots$$

where $\delta = l\,d/dx + m\,d/dy + n\,d/dz$. In the same way

$$\rho_2 = \phi + r_2\,\delta\phi + \frac{r_2^2}{1\,.\,2}\,\delta^2\phi + \ldots$$

The potential at P (as in Art. 3) is

$$V = \frac{\mu}{\kappa - 1}\,\frac{1}{E}\iint \frac{\rho_1 u_1^{\kappa-3} - \rho_2 u_2^{\kappa-3}}{u_1 - u_2}\, d\omega.$$

Substituting for ρ_1, ρ_2, this becomes

$$V = \frac{\mu}{\kappa - 1}\,\frac{1}{E}\iint\left\{\phi\,\frac{u_1^{\kappa-3} - u_2^{\kappa-3}}{u_1 - u_2} + \delta\phi\,\frac{u_1^{\kappa-4} - u_2^{\kappa-4}}{u_1 - u_2} + \ldots \frac{\delta^{\kappa-4}\phi}{L\,(\kappa - 4)}\,\frac{u_1 - u_2}{u_1 - u_2}\right.$$
$$\left. + \text{zero} + \frac{\delta^{\kappa-2}\phi}{L\,(\kappa - 2)}\,\frac{r_1 - r_2}{r_1 - r_2}(-r_1 r_2) + \frac{\delta^{\kappa-1}\phi}{L\,(\kappa - 1)}\,\frac{r_1^2 - r_2^2}{r_1 - r_2}(-r_1 r_2) + \ldots\right\} d\omega.$$

The series is divided into two parts, one with u's and the other with r's, so that all the powers used are positive. The two parts will be represented by V_1 and V_2, so that $V = V_1 + V_2$.

13. *Let us examine first the terms which contain* u *only.* These may be written in the form

$$V_1 = \frac{\mu}{\kappa - 1}\,\frac{1}{E}\iint d\omega\left\{\phi S_{\kappa-3} + \delta\phi S_{\kappa-4} + \ldots + \frac{\delta^{\kappa-3-n}\,\phi}{L\,(\kappa - 3 - n)}\,S_n + \ldots + \frac{\delta^{\kappa-4}\,\phi}{L\,(\kappa - 4)}\,S_1\right\},$$

where, as in Art. 8,

$$S_n = p^{n-1} + (n-2)\,p^{n-3}q + \ldots + \frac{L(n-f-1)}{L(f).L(n-2f-1)}\,p^{n-2f-1}q^f + \ldots$$

$$p = \frac{2F}{E} = \frac{2}{E}(\alpha xl + \beta ym + \gamma zn), \qquad q = \frac{1}{ED^2}.$$

Before attempting the general integration it is useful to examine what kind of integral we shall arrive at. We notice that every term of $\delta^{\kappa-3-n}\phi . S_n$ is an integral rational function of the direction cosines (l, m, n) of the chord. All the integrations can, therefore, be completely effected. The terms with odd powers of (l, m, n) give zero on integration, while those with even powers may be found by the theorem

$$\iint l^{2f} m^{2g} n^{2h}\, d\omega = 2\,\frac{\Gamma(f+\frac{1}{2})\,\Gamma(g+\frac{1}{2})\,\Gamma(h+\frac{1}{2})}{\Gamma(f+g+h+\frac{3}{2})}.$$

Since every term of V_1 is of the same dimensions in (l, m, n), viz., $\kappa - 4$ dimensions, we notice that the denominator of this expression, viz., $\Gamma(\kappa - 4 + \frac{3}{2})$, is the same for every term.

Supposing the density $\phi(\xi, \eta, \zeta)$ to be a homogeneous function of (ξ, η, ζ) of i dimensions, we see that V_1 will take the general form

$$V_1 = \frac{H_{\kappa-4+i}}{E^{\kappa-3}} + \frac{H_{\kappa-6+i}}{E^{\kappa-4}} + \frac{H_{\kappa-8+i}}{E^{\kappa-5}} + \ldots$$

where H_s is a homogeneous function of (x, y, z) of s dimensions. If $i > \kappa - 4$, the series ends when the exponent of E is unity, if $i < \kappa - 4$ the series ends when the numerator reduces to a constant.

Referring again to the general expression for the potential ν, we see that where the highest power in $\phi(x, y, z)$ is less than $\kappa - 2$, the only terms in the potential are those with u. It follows that *when the law of force is an even inverse* κ^{th} *power of the distance* ($\kappa > 2$), *the potential of a heterogeneous thin homœoid can be found free from signs of integration, provided the law of density at* (ξ, η, ζ) *is a function of* (ξ, η, ζ) *of less than* $\kappa - 2$ *dimensions.*

14. Let us now consider the general integration of the expression for V_1. The following lemma will be required.

Beginning (as in Art. 8) with the equation

$$\iint (\alpha xl + \beta ym + \gamma zn)^{2s}\, d\omega = (\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)^s\, \frac{4\pi}{2s+1}$$

w' ~re the integration extends all round the origin, let

$$\nabla = \frac{1}{\alpha}\frac{d^2}{dx^2} + \frac{1}{\beta}\frac{d^2}{dy^2} + \frac{1}{\gamma}\frac{d^2}{dz^2},$$

$$\nabla' = \frac{A}{\alpha}\frac{d}{dx} + \frac{B}{\beta}\frac{d}{dy} + \frac{C}{\gamma}\frac{d}{dz}$$

where A, B, C are three arbitrary constants.

We then find

$$\iint \frac{(\alpha x l + \beta y m + \gamma z n)^{2s-2f-g}}{\mathrm{L}\,(2s - 2f - g)} (\alpha l^2 + \beta m^2 + \gamma n^2)^f (\mathrm{A} l + \mathrm{B} m + \mathrm{C} n)^g \, d\omega = 4\pi \nabla^f \nabla'^g \frac{(\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)^s}{\mathrm{L}\,(2s + 1)}$$

where, as before $\mathrm{L}\,(s) = 1.2.3 \ldots s$. In this equation all the exponents are to be positive.

To apply this lemma we let A, B, C signify $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$, which operate solely on the ϕ and not on the other x, y, z which occur in the equation. The equation may then be written

$$\iint \frac{\mathrm{F}^{2s-2f-g}}{\mathrm{L}\,(2s - 2f - g)} \left(\frac{1}{\mathrm{D}^2}\right)^f \delta^g \, d\omega = 4\pi \nabla^f \nabla'^g \frac{(\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)^s}{\mathrm{L}\,(2s + 1)}$$

We see by what precedes that it will be convenient to arrange the expression for V_1 in a series of inverse powers of E. Let us, therefore, seek the term containing $\left(\frac{1}{\mathrm{E}}\right)^{h+1}$. Putting $n - f - 1 = h$, we notice that $\frac{\mathrm{S}_n \, \delta^{\kappa - 3 - n}}{\mathrm{L}\,(\kappa - 3 - n)}$ contributes the one term

$$\frac{\mathrm{L}\,(h)}{\mathrm{L}\,(n - h - 1)} \frac{(2\mathrm{F})^{2h - n + 1}}{\mathrm{L}\,(2h - n + 1)} \left(\frac{1}{\mathrm{D}^2}\right)^{n - h - 1} \frac{\delta^{\kappa - 3 - n}}{\mathrm{L}\,(\kappa - 3 - n)} \frac{1}{\mathrm{E}^h},$$

where every exponent is positive. To find all the terms with the given power of E we sum this expression, beginning at $n = h + 1$ and ending at the lesser of the two $n = 2h + 1$ or $n = \kappa - 3$.

Using the lemma, the corresponding term of V_1 becomes

$$\frac{2\pi\mu}{\kappa - 1} \mathrm{L}\,(h) \Sigma \frac{(\frac{1}{2}\Delta)^{n - h - 1}}{\mathrm{L}\,(n - h - 1)} \frac{\Delta'^{\kappa - 3 - n}}{\mathrm{L}\,(\kappa - 3 - n)} \frac{(\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)^{\frac{1}{2}(\kappa - 4)}}{\mathrm{L}\,(\kappa - 3)} \left(\frac{2}{\mathrm{E}}\right)^{h+1},$$

where Σ implies a summation with regard to n.

If $2h + 1 > \kappa - 3$ the summation extends from $n = h + 1$ to $\kappa - 3$ and the several terms form a complete binomial series. If $2h + 1 < \kappa - 3$ the summation extends from $n = h + 1$ to $2h + 1$. In this case the missing operators required to complete the binomial series are $\Delta^{h+1} \Delta'^{\kappa - 5 - 2h}$, $\Delta^{h+2} \Delta'^{\kappa - 6 - 2h}$, &c.

Since Δ, Δ' are differential operators of the orders d^2/dx^2 and d/dx, while the subject is of the order $x^{\kappa - 4}$, the missing operators reduce the subject to zero. We may, therefore, suppose the binomial series to be complete.

It immediately follows that

$$\mathrm{V}_1 = \frac{2\pi\mu}{(\kappa - 1)\,\mathrm{L}\,(\kappa - 3)} \Sigma \frac{\mathrm{L}\,(h)}{\mathrm{L}\,(\kappa - 4 - h)} \cdot \left(\frac{2}{\mathrm{E}}\right)^{h+1} (\Delta' + \tfrac{1}{2}\Delta)^{\kappa - 4 - h} \mathrm{P}\phi,$$

where (as in Art. 8),

$$P = (\alpha^2 x^2 + \beta^2 y^2 + \gamma^2 z^2)^{\frac{1}{2}(\kappa - 4)},$$

and the Σ implies summation from $h = 0$ to $h = \kappa - 4$.

When the ellipsoidal shell is homogeneous we omit the operator Δ', and put the density ϕ equal to unity. The result then agrees with that already obtained in Art. 8.

The differentiations with regard to x, y, z, which are implied by ∇, ∇', operate only on P, while the differentiations, A, B, C, in ∇' operate only on ϕ. When written at length the operator takes the form,

$$(\tfrac{1}{2}\nabla + \nabla')^s \, P\phi = \phi(\tfrac{1}{2}\nabla)^s + s\left(\frac{1}{\alpha}\frac{d\phi}{dx}\frac{d}{dx} + \frac{1}{\beta}\frac{d\phi}{dy}\frac{d}{dy} + \ldots\right)(\tfrac{1}{2}\nabla)^{s-1}P$$

$$+ s\frac{s-1}{2}\left(\frac{1}{\alpha^2}\frac{d^2\phi}{dx^2}\frac{d^2}{dx^2} + \ldots + \frac{2}{\alpha\beta}\frac{d^2\phi}{dx\,dy}\frac{d^2}{dx\,dy} + \ldots\right)(\tfrac{1}{2}\nabla)^{s-2}P + \ldots$$

15. We may write the result in a different form so as to avoid having two kinds of operators. We have

$$\nabla + 2\nabla' = \frac{1}{\alpha}\left(\frac{d}{dx} + A\right)^2 + \frac{1}{\beta}\left(\frac{d}{dy} + B\right)^2 + \frac{1}{\gamma}\left(\frac{d}{dz} + C\right)^2 - \frac{A^2}{\alpha} - \frac{B^2}{\beta} - \frac{C^2}{\gamma}.$$

Let ∇_t represent the total operator ∇ when the differentiations implied act on both P and ϕ. Let ∇_ϕ be the partial operator ∇ when the differentiations act on ϕ only, then

$$\nabla + 2\nabla' = \nabla_t - \nabla_\phi .$$

The potential of the heterogeneous homœoid therefore becomes

$$V_1 = \frac{2\pi\mu}{(\kappa - 1)\,L(\kappa - 3)}\,\Sigma\,\frac{L(h)}{L(\kappa - 4 - h)}\left(\frac{2}{E}\right)^{h+1}(\nabla_t - \nabla_\phi)^{\kappa-4-h}\,P\phi\,(\tfrac{1}{2})^{\kappa-4-h}.$$

16. As an example, let us find the potential of a thin homœoid at an internal point (x, y, z) when the force varies as the inverse fourth power and the density at (ξ, η, ζ) is $m\xi$.

The series has only one term given by $h = 0$, since $\kappa = 4$. The result is seen at a glance to be

$$V = \frac{2\pi\mu m}{3}\frac{2}{E}x .$$

Thus the potential is the same as if the homœoid were homogeneous and its density were mx.

This curious result may be verified by independent reasoning. Let P be the given internal point, PQ any radius vector of the shell. Let the direction cosines of PQ be (l, m, n) and its length be r. The density at Q is therefore proportional to $(x + lr)$. The potential is

$$V = \frac{x}{3}\iint\frac{\mu p d\sigma}{r^3} + \frac{1}{3}\iint\frac{\mu p l d\sigma}{r^2}.$$

The first term is the potential of the homogeneous shell. The second is the x attraction of a homogeneous homœoid where the law of force is the inverse square and is therefore zero.

As another example, let us find the potential of a heterogeneous thin homœoid when the force varies as the inverse sixth power and the density at (ξ, η, ζ) is $m\xi^2$.

The series has three terms given by $h = 0, 1, 2$. Hence, since $\kappa = 6$,

$$V = \frac{\pi\mu}{15}\left\{\frac{2}{E}\frac{1}{2}\left(\nabla' + \frac{\nabla}{2}\right)^2 + \frac{4}{E^2}\left(\nabla' + \frac{\nabla}{2}\right) + \frac{8}{E^3}2\right\}P\phi.$$

Now $\nabla P = 2(\alpha + \beta + \gamma)$, $\nabla^2 P = 0$, $\nabla\nabla' P = 0$, $\nabla' P = 2\alpha A x + \&c.$, $\nabla'^2 P = 2A^2 + \&c.$ Remembering that $\phi = mx^2$ and $A\phi = d\phi/dx$, $A^2\phi = d^2\phi/dx^2$, we find, after an easy reduction,

$$V = \frac{4\pi\mu m}{15}\left\{\frac{1}{E} + \frac{(5\alpha + \beta + \gamma)x^2}{E^2} + \frac{4(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)x^2}{E^3}\right\}.$$

It immediately follows that the potential at an external point x', y', z' for the same law of density is

$$V' = \frac{4\pi\mu m}{15}\frac{abc}{a'b'c'}\frac{a^2}{a'^2}\left\{\frac{1}{E'} + \left(\frac{5}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}\right)\frac{a^2x'^2}{a'^2}\frac{1}{E'^2} + \left(\frac{a^2x'^2}{a'^6} + \frac{b^2y'^2}{b'^6} + \frac{c^2z'^2}{c'^6}\right)\frac{a^2x'^2}{a'^2}\frac{4}{E'^3}\right\},$$

where

$$E' = \epsilon^2\left(\frac{x'^2}{a'^4} + \frac{y'^2}{b'^4} + \frac{z'^2}{c'^4}\right) = \frac{\epsilon^2}{p'^2},$$

and a', b', c' are the semi-axes of the confocal through the attracted point.

17. As a third example, we may apply the formula to find the potential of a thin homogeneous focaloid when κ is greater than 4.

The thickness of a thin homœoid is μp, while that of a thin focaloid varies inversely as p. A homœoid, whose density is m/p^2, is equivalent to a homogeneous focaloid of unit density whose $ada = m\mu$. Putting then

$$\phi = m/p^2 = m(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2),$$

we have, if $n = \kappa - 4 - h$,

$$(\nabla_l - \nabla_\phi)^n \mathrm{P}\phi = \{\nabla_l^n - n\nabla_l^{n-1}\nabla_\phi\}\,\mathrm{P}\phi$$
$$= \nabla^n(\mathrm{P}\phi) - 2n(\alpha+\beta+\gamma)\,m\nabla^{n-1}\mathrm{P}$$
$$= m\nabla^n\left(\frac{1}{p^2}\right)^{\frac{1}{2}(\kappa-2)} - 2n(\alpha+\beta+\gamma)\,m\nabla^{n-1}\left(\frac{1}{p^2}\right)^{\frac{1}{2}(\kappa-4)}.$$

Substituting this in the formula for V, already written down, we have the required potential.

Suppose the law of force to be the inverse sixth power, the potential of a homogeneous thin focaloid, at an internal point, is

$$\mathrm{V} = \frac{4\pi\mu m}{15}\left\{\frac{\alpha^2+\beta^2+\gamma^2}{\mathrm{E}} + \frac{1}{\mathrm{E}^2}\left(\frac{\alpha+\beta+\gamma}{p^2} + 4(\alpha^3x^2+\beta^3y^2+\gamma^3z^2)\right) + \frac{4}{\mathrm{E}^3}\frac{1}{p^4}\right\},$$

and at an external point

$$\mathrm{V}' = \frac{4\pi\mu m}{15}\,\frac{abc}{a'b'c'}\,\frac{1}{\epsilon^4}\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{4}{\epsilon^2}\right)p'^2,$$

where a', b', c' are the axes of the confocal through the external point, and $\epsilon^2 = a'^2 - a^2$. It appears that the potential varies as the square of the perpendicular p' on the tangent plane for all points on the same confocal.

18. When the density $\phi(\xi, \eta, \zeta)$ of the thin homœoid is a function of the coordinates of a degree higher than $\kappa - 2$, the potential includes some of the terms of Art. 12, which contain r_1 and r_2. We shall now examine these terms. In this way we shall arrive at an expression for *the potential of a thin heterogeneous homœoid at an internal point where the index κ of the force is a positive even integer greater than the square.*

These additional terms are given by

$$\mathrm{V}_2 = \frac{\mu}{\kappa-1}\iint\left[\frac{\delta^{\kappa-2}\phi}{\mathrm{L}(\kappa-2)}\mathrm{S}_1 + \frac{\delta^{\kappa-1}\phi}{\mathrm{L}(\kappa-1)}\mathrm{S}_2 + \ldots + \frac{\delta^{\kappa+n-3}\phi}{\mathrm{L}(\kappa+n-3)}\mathrm{S}_n + \ldots\right]\left(\frac{-r_1r_2}{\mathrm{E}}\right)d\omega,$$

$$\mathrm{S}_n = \frac{r_1^n - r_2^n}{r_1 - r_2} = p^{n-1} + (n-2)p^{n-3}q + \ldots + \frac{\mathrm{L}(n-f-1)}{\mathrm{L}(f)\,.\,\mathrm{L}(n-2f-1)}p^{n-2f-1}q^f + \ldots,$$

$$p = -2\mathrm{D}^2\mathrm{F}, \qquad q = -r_1r_2 = \mathrm{ED}^2,$$

and D, E, F are expressed as functions of the direction cosines (l, m, n) in Art. 1.

Before attempting the general integration, it is convenient to examine the kind of integral we shall arrive at.

Consider first the dimensions of the several terms of V_2 as regards the coordinates (x, y, z) of the attracted point. Let $\phi(x, y, z)$ be homogeneous and of i dimensions, then remembering that F is a linear function of (x, y, z) we see that, if the terms are arranged in powers of q or E, V_2 takes the form

$$\mathrm{V}_2 = \mathrm{H}_{i-\kappa+2} + \mathrm{EH}_{i-\kappa} + \mathrm{E}^2\mathrm{H}_{i-\kappa-2} + \ldots$$

where H_s is a homogeneous function of (x, y, z) of s dimensions. The series terminates when H_s reduces to a constant.

To calculate the values of H_s we must consider the dimensions of V_2 with regard to (l, m, n). Since positive powers of D now occur in the series, the integrations cannot be completely effected. The coefficients of the powers of (x, y, z) in H_s will be definite integrals of the same kind as those which occur when the force varies as the inverse square.

The terms of V_2 which contain any odd exponents of (l, m, n) give zero after integration. Omitting these it will be seen that every term of V_2 is of the form $l^{2f}m^{2g}n^{2h} D^{2(f+g+h)-(\kappa-4)}$, where f, g, h here stand for any positive integers. Now putting $t = \frac{1}{2}(\kappa - 2)$, the integral of any one of these terms may be found by the equation

$$\iint \frac{l^{2f}m^{2g}n^{2h}\, d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{s+1-t}} = N\left(\frac{d}{d\alpha}\right)^f\left(\frac{d}{d\beta}\right)^g\left(\frac{d}{d\gamma}\right)^h \int_0^\infty \frac{v^{t-\frac{1}{2}}\, dv}{Q},$$

where

$$N = \frac{2\pi L(t)}{L(2t)} 2^{2t} \frac{(-1)^s}{L(s-t)}, \qquad s = f + g + h,$$

$$Q^2 = (\alpha + v)(\beta + v)(\gamma + v).$$

In any particular example, therefore, the integrations may be effected (as far as they can be done) by using this equation.

19. To obtain the general integral of the series for V_2, we require a lemma.

We deduce from the integral given in the last article that

$$\iint \frac{(\lambda l + \mu m + \nu n)^{2s}\, d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{s+1-t}} = K\int_0^\infty \left(\frac{\lambda^2}{\alpha + v} + \frac{\mu^2}{\beta + v} + \frac{\nu^2}{\gamma + v}\right)^s \frac{v^{t-\frac{1}{2}}\, dv}{Q},$$

where

$$K = \frac{L(2s)}{L(s)} \frac{1}{L(s-t)} \frac{2\pi}{2^{2s-2t}} \frac{L(t)}{L(2t)},$$

$$Q^2 = (\alpha + v)(\beta + v)(\gamma + v),$$

and t is a positive integer less than s (see the author's 'Statics,' 1892, vol. 2, Arts. 219 and 220 for this integral and the one used in the last article).

Let

$$\Delta = A\frac{d}{d\lambda} + B\frac{d}{d\mu} + C\frac{d}{d\nu},$$

where A, B, C are any constants, we find

$$\iint \frac{(\lambda l + \mu m + \nu n)^{2s-g}(Al + Bm + Cn)^g\, d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{s+1-t}}$$

$$= \frac{L(2s-g)}{L(2s)} K\, \Delta^g \int_0^\infty \left\{\frac{\lambda^2}{\alpha + v} + \&c.\right\}^s \frac{v^{t-\frac{1}{2}}\, dv}{Q}.$$

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To apply this lemma we suppose, as before, that A, B, C represent differentiations with regard to x, y, z, but to operate only on ϕ. Hence supposing λ, μ, ν to stand for αx, βy, γz, and using the abbreviations given in Art. 1, we write the lemma in the form

$$\iint F^{2s-g} (D^2)^{s+1-t} \delta^g \, d\omega = \frac{L(2s-g)}{L(2s)} K \Delta^g \int_0^\infty \left\{ \frac{\lambda^2}{\alpha + v} + \&c. \right\}^s \frac{v^{t-\frac{1}{2}} dv}{Q}.$$

20. By substituting for S_n its value we find

$$\iint \frac{\delta^{\kappa+n-3}\phi}{L(\kappa+n-3)} S_n D^2 \, d\omega = \iint \frac{\delta^{\kappa+n-3}\phi}{L(\kappa+n-3)} \Sigma \frac{L(n-f-1)}{Lf . L(n-2f-1)} (-2F)^{n-2f-1} D^{2(n-f)} E^f.$$

To use the lemma we put $g = \kappa + n - 3$, $2s - g = n - 2f - 1$, $s + 1 - t = n - f$. These give $t = \frac{1}{2}(\kappa - 2)$ so that κ must not be less than 2. They also make $s - t$ positive. The double integral therefore becomes

$$\frac{(-1)^{n-1}}{L(n+2t-1)} \frac{2\pi}{2^{n-1}} \frac{L(t)}{L(2t)} \Delta^{n+2t-1} \int_0^\infty \Sigma \frac{E^f}{L(f) L(s)} \left(\frac{\lambda^2}{\alpha+v} + \dots \right)^s \frac{v^{t-\frac{1}{2}} dv}{Q},$$

where $2(t+1)$ has been written for κ and $s = n - f + t - 1$.

Since Σ here implies summation from $f = 0$ to $f = \frac{1}{2}n - 1$ or $\frac{1}{2}(n-1)$, the terms which follow the Σ do not form a complete binomial expansion. But, since the operator Δ^{n+2t-1} contains differential coefficients of the order $(d/d\lambda)^{n+2t-1}$, while the subject is of the order $\lambda^{2(n-f+t-1)}$, the missing terms are zero. We may therefore sum the series from $f = 0$ to $n + t - 1$. We therefore have

$$V_2 = \frac{2\pi\mu}{\kappa - 1} \cdot \frac{L(t)}{L(2t)} \cdot \Sigma \int_0^\infty \frac{\Delta^{n+2t-1}}{L(n+2t-1)} \frac{R^{n+t-1}}{L(n+t-1)} (-\tfrac{1}{2})^{n-1} \phi \frac{v^{t-\frac{1}{2}} dv}{Q},$$

where

$$R = E + \frac{\lambda^2}{\alpha+v} + \frac{\mu^2}{\beta+v} + \frac{\nu^2}{\gamma+v}$$

and Σ implies summation for integer values of n, beginning at $n = 1$ onwards until the terms vanish by the operation of the differential coefficients A, B, C, contained in Δ on the subject $\phi(x, y, z)$.

21. We may simplify this series by performing the operation $\Delta^{n+2t-1} R^{n+t-1}$ once for all. For the sake of brevity let

$$S = 2\left(\frac{A\lambda}{\alpha+v} + \frac{B\mu}{\beta+v} + \frac{C\nu}{\gamma+v} \right) \qquad T = \frac{A^2}{\alpha+v} + \frac{B^2}{\beta+v} + \frac{C^2}{\gamma+v}.$$

We then find, when $g < h$,

$$\frac{\Delta^g R^h}{L(g) . L(h)} = \frac{R^{h-g} S^g}{L(h-g) . L(g)} + \frac{R^{h-g+1} S^{g-2} T}{L(h-g+1) . L(g-2)} + \frac{R^{h-g+2} S^{g-4} T^2}{L(h-g+2) . L(g-4) . L(2)} + \&c.,$$

the series stopping when the exponent of S becomes negative. The form of the expansion having been found, the simplest proof is that supplied by the method of induction. Performing Δ on both sides of this equation and noticing that

$$\Delta R^m S^n = mR^{m-1}S^{n+1} + nR^m S^{n-1} \,.\, 2T,$$

we may see that since the theorem is true for $g = 0$, it is true for all positive integer values of g. It is unnecessary to reproduce the work.

When $g > h$, some of the indices of R at the beginning of the series are negative. These terms are to be omitted and we then have

$$\frac{\Delta^g R^h}{L(g)L(h)} = \frac{S^{2h-g}T^{g-h}}{L(2h-g).L(g-h)} + \frac{RS^{2h-g-2}T^{g-h+1}}{L(2h-g-2)L(g-h+1)} + \frac{R^2 S^{2h-g-4}T^{g-h+2}}{L(2)L(2h-g-4)L(g-h+2)} + \&c.$$

We thus have

$$\frac{\Delta^{n+2t-1} R^{n+t-1}}{L(n+2t-1)\,L(n+t-1)}\left(-\tfrac{1}{2}\right)^{n-1} = \frac{(-\frac{1}{2}S)^{n-1}T^t}{L(n-1)\,L(t)} + \frac{R(-\frac{1}{2}S)^{n-3}T^{t+1}}{L(n-3)\,L(t+1)}\,\frac{1}{2^2} + \ldots$$

In this equality S and T contain A, B, C, which represent differential coefficients of the function $\phi\,(x, y, z)$. Also to find V_2 we have to sum both sides of this equality for all values of n from $n = 1$ onwards, until the terms become zero by the continued operation of S and T on ϕ. Since the series stops at the first negative power of S, all negative exponents are to be omitted. We therefore write the summation in the form

$$\Sigma \frac{\Delta^{n+2t-1}R^{n+t-1}}{L(n+2t-1)\,L(n+t-1)}(-\tfrac{1}{2})^{n-1}\phi = \frac{T^t}{L(t)}\,\Sigma\frac{(-\frac{1}{2}S)^{n-1}\phi}{L(n-1)} + \frac{RT^{t+1}}{L(t+1)}\,\frac{1}{2^2}\,\Sigma\frac{(-\frac{1}{2}S)^{n-3}\phi}{L(n-3)} + \ldots$$

We notice that on the right-hand side the first Σ implies summation from $n = 1$ to ∞, the second from $n = 3$ to ∞, and so on. Now by TAYLOR's theorem

$$e^{-\frac{1}{2}S}\phi = \Sigma\frac{(-\frac{1}{2}S)^h\phi}{L(h)} = \phi\left(x - \frac{\lambda}{\alpha + v},\quad y - \frac{\mu}{\beta + v},\quad z - \frac{\nu}{\gamma + v}\right)$$

where the Σ implies summation from $h = 0$ to ∞. We therefore have

$$V_2 = \frac{2\pi\mu}{\kappa - 1}\,\frac{L(t)}{L(2t)}\int_0^\infty \frac{v^{t-\frac{1}{2}}\,dv}{Q}\left[\frac{T^t}{L(t)} + \frac{RT^{t+1}}{L(t+1)}\,\frac{1}{2^2} + \ldots\right]\phi\left(x - \frac{\lambda}{\alpha + v}, \&c.\right).$$

Here T is an operator on (x, y, z). The quantities λ, μ, ν are constants for which we write $\alpha x, \beta y, \gamma z$ after the operations indicated by T have been performed. We may reverse the order of these operations and write

$$x - \frac{\lambda}{\alpha + v} = \frac{vx}{\alpha + v}, \quad y - \frac{\mu}{\beta + v} = \frac{vy}{\beta + v}, \text{ \&c.,}$$

before the operation T has been performed, provided we replace the d/dx, d/dy, d/dz included in T by $\frac{\alpha + v}{v}\frac{d}{dx}$, $\frac{\beta + v}{v}\frac{d}{dy}$, $\frac{\gamma + v}{v}\frac{d}{dz}$ respectively.

The operator T then takes the form

$$T = \frac{1}{v^2}\left\{(\alpha + v)\frac{d^2}{dx^2} + (\beta + v)\frac{d^2}{dy^2} + (\gamma + v)\frac{d^2}{dz^2}\right\};$$

also by the same substitution

$$R = 1 - \frac{\alpha v x^2}{\alpha + v} - \frac{\beta v y^2}{\beta + v} - \frac{\gamma v z^2}{\gamma + v}$$

which must now be placed on the left-hand side of the operator to indicate that it is not included in the subject.

We thus find for the partial potential of a thin heterogeneous homœoid, whose density at the point (ξ, η, ζ) is $\phi(\xi, \eta, \zeta)$,

$$V_2 = \frac{2\pi\mu}{\kappa - 1}\frac{L(t)}{L(2t)}\int_0^\infty \frac{v^{t-\frac{1}{2}}dv}{Q}\left\{\frac{T^t}{L(t)} + \frac{RT^{t+1}}{L(t+1)}\frac{1}{2^2} + \ldots\right\}\phi\left(\frac{vx}{\alpha + v}, \ldots\right).$$

To this we add the partial potential V_1, already found in Art. 14, for the complete potential.

22. It is sometimes more convenient to use the axes a, b, c of the ellipsoid instead of the squared reciprocals α, β, γ. Let us also suppose that the density at any point (ξ, η, ζ) is given by $\psi(\xi/a, \eta/b, \zeta/c)$, so that we may the more easily deduce the potential at an external point. Also writing $v = 1/u$, we have

$$R = 1 - \frac{x^2}{a^2 + u} - \frac{y^2}{b^2 + u} - \frac{z^2}{c^2 + u},$$

$$D = \frac{T}{u} = \frac{a^2 + u}{a^2}\frac{d^2}{dx^2} + \frac{b^2 + u}{u^2}\frac{d^2}{dy^2} + \frac{c^2 + u}{c^2}\frac{d^2}{dz^2},$$

$$Q_1^2 = (a^2 + u)(b^2 + u)(c^2 + u).$$

The partial potential of a thin heterogeneous homœoid at an internal point is then

$$V_2 = \frac{2\pi\mu}{\kappa - 1}\frac{L(t)}{L(2t)}\int_0^\infty \frac{abc\,du}{Q_1} M\psi\left(\frac{ax}{a^2 + u}, \frac{by}{b^2 + u}, \frac{cz}{c^2 + u}\right),$$

where the operator M *is*

$$M = \frac{D^t}{L(t)} + \frac{uRD^{t+1}}{L(t+1)}\frac{1}{2^2} + \ldots + \frac{u^n R^n D^{t+n}}{L(n) \,.\, L(t+n)}\frac{1}{2^{2n}} + \ldots,$$

and $t = \frac{1}{2}(\kappa - 2)$ *where* κ *is the index of the law of force.*

Having found the law of the potential of a homœoid for the inverse κ^{th} power, the potential for the inverse $(\kappa + 2)^{\text{th}}$ power may be obtained by JELLETT's theorem, already mentioned in Art. 5. We might thus obtain a new proof of the formula by the method of induction. The theorem has been verified in this way, but the demonstration is long and rather tedious.

23. To deduce the potential of the heterogeneous homœoid at an external point (x', y', z'), we follow the same process as before. Let a', b', c' be the axes of the confocal through the attracted point. Let $a'^2 - a^2 = \epsilon^2$, so that ϵ^2 is determined by

$$\frac{x'^2}{a^2 + \epsilon^2} + \frac{y'^2}{b^2 + \epsilon^2} + \frac{z'^2}{c^2 + \epsilon^2} = 1.$$

We now write a', b', c' and μ' for a, b, c and μ, where $\mu' \; a'b'c' = \mu \; abc$, since the volumes of the two shells are to be equal. Lastly, putting $u + \epsilon^2 = w$, $x/a = x'/a'$, &c., we find after some easy reductions that the *partial potential of the heterogeneous homœoid at an external point is*

$$V'_2 = \frac{2\pi\mu}{\kappa - 1}\frac{L(t)}{L(2t)}\int_{\epsilon^2}^{\infty}\frac{abc\,dw}{Q_1}\,M\psi\left(\frac{ax'}{a^2 + w}, \frac{by'}{b^2 + w}, \frac{cz'}{c^2 + w}\right),$$

$$uR = wR' = w\left\{1 - \frac{x'^2}{a^2 + w} - \frac{y'^2}{b^2 + w} - \frac{z'^2}{c^2 + w}\right\},$$

$$D' = \frac{a^2 + w}{a^2}\frac{d^2}{dx'^2} + \frac{b^2 + w}{b^2}\frac{d^2}{dy'^2} + \frac{c^2 + w}{c^2}\frac{d^2}{dz'^2},$$

$$M = \frac{D'^t}{L(t)} + \frac{wR'D'^{t+1}}{L(t+1)}\frac{1}{2^2} + \ldots + \frac{w^n R'^n D'^{t+n}}{L(n)\,L(t+n)}\frac{1}{2^{2n}} + \ldots$$

24. As a first example, *let the law of force be the inverse square.* We then have $\kappa = 2$ and therefore $t = 0$. The partial potential V_1 is then zero and the complete potentials, inside and outside, are given by the expressions for V_2 and V_2' just written down. These agree with those given by Mr. DYSON in the 'Quarterly Journal,' vol. 25, 1891.

As a second example, let us find the potential of a thin homœoid when the *law of force is the inverse fourth power and the density* at (ξ, η, ζ) *is* $m\,(\xi/a^2)$. We then find by Arts. (14) and (22),

$$V_1 = \frac{2\pi\mu}{3}\frac{2}{E}\,m\left(\frac{x}{a}\right)^2, \qquad V_2 = \frac{2\pi\mu}{3}\int_0^{\infty}\frac{abc\,du}{Q_1}\frac{m}{a^2 + u}.$$

The potential at an internal point is the sum of these two.

25. *To find the potential of a thin heterogeneous homœoid when the force varies as the inverse κ^{th} power of the distance, and $\kappa = 2$, zero, or any even negative number.*

As the investigation resembles that of the last proposition, except as regards the lemma, it is sufficient merely to sketch the argument.

To avoid confusion as to the sign of κ, let us, in the first instance, put $\kappa = -\kappa_1$ We then have, by Art. 2,

$$V = -\frac{\mu}{\kappa_1 + 1}\iint \frac{\rho_1 r_1^{\kappa_1+3} - \rho_2 r_2^{\kappa_1+3}}{r_1 - r_2} D^2 d\omega\,;$$

writing as before (Art. 12),

$$\rho = \phi + r\delta\phi + r^2\frac{\delta^2\phi}{1.2} + \ldots$$

$$V = -\frac{\mu}{\kappa_1 + 1}\iint\left\{\phi S_{\kappa_1+3} + \delta\phi S_{\kappa_1+4} + \ldots\right\} D^2 d\omega$$

$$= -\frac{\mu}{\kappa_1 + 1}\Sigma\iint \frac{\delta^{n-\kappa_1-3}\phi}{L(n - \kappa_1 - 3)} S_n D^2 d\omega,$$

where Σ implies summation from $\kappa_1 + 3$ to ∞. We now substitute, as before,

$$S_n = \frac{r_1^n - r_2^n}{r_1 - r_2} = \Sigma\frac{L(n - f - 1)}{L(f)L(n - 2f - 1)} p^{n-2f-1} q^f,$$

where $p = -2D^2F$, $q = ED^2$.

The lemma by which the integration of each term is effected is as follows. By Art. 10 we have

$$\iint \frac{(\lambda l + \mu m + \nu n)^{2s} d\omega}{(\alpha l^2 + \beta m^2 + \gamma n^2)^{s+t+1}} = K' . \int_0^\infty v^{-\frac{1}{2}} dv \left(\frac{d}{dv}\right)^t \left\{\frac{\lambda^2}{\alpha + v} + \frac{\mu^2}{\beta + v} + \frac{\nu^2}{\gamma + v}\right\}^s \frac{1}{Q}.$$

Following the same notation as before, we let

$$\Delta = A\, d/d\lambda + B\, d/d\mu + C\, d/d\nu.$$

We then find, as in Art. 19,

$$\iint F^{2s-g}(D^2)^{s+t+1}\delta^g d\omega = \frac{L(2s - g)}{L(s)\,L(s + t)}\frac{2\pi(-1)^t}{2^{2s}}\Delta^g\int_0^\infty v^{-\frac{1}{2}} dv\left(\frac{d}{dv}\right)^t \{\&c.\}^s \frac{1}{Q},$$

where g is positive and less than $2s$.

To apply the lemma we put $g = n - \kappa_1 - 3$, $2s - g = n - 2f - 1$, $s + t = n - f - 1$. These give $t = \frac{1}{2}(\kappa_1 + 2) = \frac{1}{2}(2 - \kappa)$ and this requires that the index κ of the force should be less than 2.

After completing and summing the same binomial series as before, we have

$$V = -\frac{2\pi\mu}{\kappa_1 + 1}\Sigma(-1)^{n-t-1}\frac{(\frac{1}{2}\Delta)^{n-2t-1}\phi}{L(n-2t-1)}\int_0^\infty v^{-\frac{1}{2}}\,dv\left(\frac{d}{dv}\right)^t\frac{R^{n-t-1}}{L(n-t-1)}\frac{1}{Q},$$

where, as in Art. 20,

$$R = E + \frac{\lambda^2}{\alpha + v} + \frac{\mu^2}{\beta + v} + \frac{\nu^2}{\gamma + v}.$$

By using the theorem of Art. 21, we can perform the operations indicated by Δ. Noticing that the index of the operator (viz., $n - 2t - 1$) is here less than that of the subject (viz., $n - t - 1$), we find

$$V = -\frac{2\pi\mu}{\kappa_1 + 1}(-1)^t\int_0^\infty v^{-\frac{1}{2}}\,dv\left(\frac{d}{dv}\right)^t M\frac{1}{Q}\phi\left(x - \frac{\lambda}{\alpha + v},\ y - \frac{\mu}{\beta + v},\ \&c.\right),$$

$$M = \frac{R^t}{L(t)} + \frac{R^{t+1}T}{L(t+1)}\frac{1}{2^2} + \ldots\frac{R^{t+n}T^n}{L(t+n)L(n)}\frac{1}{2^{2n}} + \ldots$$

We now write $\lambda = \alpha x$, &c., and make the necessary changes in the meanings of R and T. We thus have

$$T = \frac{1}{v^2}\left\{(\alpha + v)\frac{d^2}{dx^2} + (\beta + v)\frac{d^2}{dy^2} + (\gamma + v)\frac{d^2}{dz^2}\right\},$$

$$R = 1 - \frac{\alpha v x^2}{\alpha + v} - \frac{\beta v y^2}{\beta + v} - \frac{\gamma v z^2}{\gamma + v}.$$

26. We may conveniently express the result in terms of the axes of a, b, c of the ellipsoid instead of the squared reciprocals. We find that, when the index κ of the force is an even integer < 2 or negative, the potential of a thin heterogeneous homœoid, the density at (ξ, η, ζ) being $\psi(\xi/a, \eta/b, \zeta/c)$ at an internal point, is

$$V = \frac{2\pi\mu}{\kappa - 1}\int_0^\infty u^{-\frac{3}{2}}\,du\left(u^2\frac{d}{du}\right)^t\frac{abc\,u^{\frac{3}{2}}}{Q_1}M\psi\left(\frac{ax}{a^2 + u},\ \frac{by}{b^2 + u},\ \frac{cz}{c^2 + u}\right),$$

where

$$M = \frac{R^t}{L(t)} + \frac{R^{t+1}}{L(t+1)}uD.\frac{1}{2^2} + \ldots + \frac{R^{t+n}}{L(t+n)}\frac{(uD)^n}{L(n)}\frac{1}{2^{2n}} + \ldots,$$

and R, D, and Q_1 have the meanings given to them in Art. 22. Also $t = \frac{1}{2}(2 - \kappa)$.

At an external point the potential is

$$V' = \frac{2\pi\mu}{\kappa - 1}\int_{\epsilon^2}^\infty (u - \epsilon^2)^{-\frac{3}{2}}\,du\left\{(u - \epsilon^2)^2\frac{d}{du}\right\}^t\frac{abc\,u^t}{Q_1(u - \epsilon^2)^{t-\frac{3}{2}}}M\psi\left(\frac{ax}{a^2 + u},\ \&c.\right),$$

where (x, y, z) are now the coordinates of the external attracted point, so that ϵ^2 is defined by

$$\frac{x^2}{a^2 + \epsilon^2} + \frac{y^2}{b^2 + \epsilon^2} + \frac{z^2}{c^2 + \epsilon^2} = 1.$$

27. *To find the potential of a solid homogeneous ellipsoid at an internal point when the law of force is the inverse κ^{th} power and κ is even and > 4.*

Referring to Art. 4, we see that when $\kappa > 4$,

$$V = \frac{-1}{2(\kappa-1)(\kappa-4)} \iint (u_1^{\kappa-4} + u_2^{\kappa-4})\, d\omega.$$

Proceeding as in Art. 8, we have

$$\frac{u_1^n + u_2^n}{n} = \Sigma \frac{L(n-f-1)}{L(f)\,L(n-2f)} \left(\frac{2F}{E}\right)^{n-2f} \left(\frac{1}{ED^2}\right)^f,$$

where Σ implies summation from $f = 0$ to $\frac{1}{2}n$. Writing $n = \kappa - 4$, and remembering the integral given in Art. 8, we have

$$V = \frac{-2\pi}{(\kappa-1)\,L(\kappa-3)} \left(\frac{2}{E}\right)^{\kappa-4} \Sigma \frac{L(\kappa-5-f)}{L(f)} \frac{E^f \nabla^f}{2^{2f}} P,$$

where

$$P = \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)^{\frac{1}{2}(\kappa-4)},$$

$$\nabla = a^2 \frac{d^2}{dx^2} + b^2 \frac{d^2}{dy^2} + c^2 \frac{d^2}{dz^2},$$

and Σ implies summation from $f = 0$ to $\frac{1}{2}(\kappa - 4)$. Thus the series has but two terms when the law of force is the inverse sixth, three terms when the inverse eighth, and so on.

28. *To find the potential of a homogeneous solid ellipsoid at an internal point when the law of force is the inverse κ^{th} and κ is even < 4, and may be negative.*

Here

$$V = \frac{-1}{2(\kappa-1)(\kappa-4)} \iint (r_1^{4-\kappa} + r_2^{4-\kappa})\, d\omega,$$

$$\frac{r_1^n + r_2^n}{n} = \Sigma \frac{L(n-f-1)}{L(f)\,.\,L(n-2f)} (-2FD^2)^{n-2f} (ED^2)^f.$$

Substituting and using the integral form given in the lemma, Art. 10, we find, after summing the same binomial series,

$$V = \frac{\pi(-1)^t}{(\kappa-1)\,L(t+1)} \int_0^\infty v^{-\frac{1}{2}}\, dv \left(\frac{d}{dv}\right)^t \frac{R^{t+1}}{Q},$$

where

$$R = 1 - \frac{v\alpha x^2}{\alpha+v} - \frac{v\beta y^2}{\beta+v} - \frac{v\gamma z^2}{\gamma+v}$$

$$Q^2 = (\alpha+v)(\beta+v)(\gamma+v)$$

and

$$t = \tfrac{1}{2}(2-\kappa).$$

29. Let us next turn our attention to the attraction of a heterogeneous solid ellipsoid at an internal point. The density at any point x, y, z being $\phi(x, y, z)$, the density at a neighbouring point $x+\xi, y+\eta, z+\zeta$ is

$$\phi + \phi_x\xi + \phi_y\eta + \phi_z\zeta + \ldots$$

Let us describe a sphere of small radius r, whose centre is the point x, y, z. The x attraction at the centre of the matter filling this sphere, supposing the law of attraction to be the inverse κ^{th} and the density to be $m\xi$, is seen by an easy integration to be $\frac{4\pi}{3}m\left[\frac{r^{4-\kappa}}{4-\kappa}\right]_0^r$ except when $\kappa = 4$. If κ is greater than 4, this attraction is infinite. If $\kappa = 4$ the integral is $\frac{4\pi}{3}[\log r]_0^r$ and is again infinite. The attraction at any internal point of a heterogeneous ellipsoid is therefore in general infinite whenever the law of force is such that the index κ is greater than 3. We shall therefore confine our attention to those laws of force which make κ less than 4. When the ellipsoid is homogeneous the x attraction of the matter filling the elementary sphere is zero at the centre, and there is no restriction on the value of κ.

30. *To find the potential at an internal point* P *of a heterogeneous ellipsoid whose density is* $\phi(x, y, z)$, *when the law of force is the inverse* κ^{th}, *and* κ *is even and less than* 4.

Referring to Art. 4 we see that the potential at P of the conical element whose axis is in the positive direction of the radius vector is $\frac{1}{\kappa-1}\int\frac{r^2\,d\omega\,dr\rho}{r^{\kappa-1}}$, the limits being $r = 0$ to r. As in Art. 12, we write

$$\rho = \phi + r\,\delta\phi + \frac{r^2}{1.2}\delta^2\phi + \ldots + \frac{r^m}{L(m)}\delta^m\phi + \ldots$$

The potential of the positive conical element is therefore $\frac{d\omega}{\kappa-1}\Sigma\frac{r^{4-\kappa+}}{4-\kappa+m}\frac{\delta^m\phi}{L(m)}$. To find the potential of the negative cone, we let R be the distance from P of one of its elements Q′, taken positively. The density at Q′ is found by writing $-$ R for r in the above series for ρ. The potential of the negative cone is, therefore, $\frac{d\omega}{\kappa-1}\Sigma\frac{R^{4-\kappa+m}}{4-\kappa+m}(-1)^m\frac{\delta^m\phi}{L(m)}$, and here we must write $-r_2$ for R.

Taking the two conical elements together, and writing n for $4-\kappa+m$, the potential of the whole ellipsoid at P becomes

$$V = \frac{1}{2(\kappa-1)}\iint d\omega\,\Sigma\frac{r_1^n + r_2^n}{n}\frac{\delta^{n+\kappa-4}\phi}{L(n+\kappa-4)}.$$

The integral is to be taken all round the attracted point. The summation represented by Σ extends from $n = 4-\kappa$ to infinity.

The investigation now proceeds on the same general lines as in Art. 25. By Art. 8 we have

$$\frac{r_1^n + r_2^n}{n} = \Sigma \frac{\mathrm{L}(n-f-1)}{\mathrm{L}f \,.\, \mathrm{L}(n-2f)}(-2\mathrm{FD}^2)^{n-2f}(\mathrm{ED}^2)^f,$$

where the sum extends from $f = 0$ to $\frac{1}{2}n$ or $\frac{1}{2}(n-1)$. Substituting and effecting an integration by the use of the lemma of Art. 25, we find, after putting $t = \frac{1}{2}(2-\kappa)$,

$$\iint \frac{r_1^n + r_2^n}{n} \frac{\delta^{n+\kappa-4}}{\mathrm{L}(n+\kappa-4)} d\omega = \frac{(-1)^{n+t} 2\pi}{\mathrm{L}(n-2t-2)} \left(\frac{\Delta}{2}\right)^{n-2t-2} \int_0^\infty v^{-\frac{1}{2}} dv \left(\frac{d}{dv}\right)^t \cdot \Sigma \frac{\mathrm{E}^f (\mathrm{R}-\mathrm{E})^s}{\mathrm{L}(f) \,.\, \mathrm{L}(s)} \cdot \frac{1}{\mathrm{Q}},$$

where

$$\mathrm{R} = \mathrm{E} + \frac{\lambda^2}{\alpha + v} + \frac{\mu^2}{\beta + v} + \frac{\nu^2}{\gamma + v},$$

$$\Delta = \mathrm{A}\frac{d}{d\lambda} + \mathrm{B}\frac{d}{d\mu} + \mathrm{C}\frac{d}{d\nu} \cdot$$

The summation implied by the Σ does not make a complete binomial series. But since the operator contains differential coefficients of the order $n - 2t - 2$ while the subject is of the order $2s$ where $s = n - f - t - 1$, the terms given by any value of f greater than $\frac{1}{2}n$ are zero. The missing terms of the binomial series are therefore zero. Summing the series we find

$$\mathrm{V} = \Sigma \frac{\pi(-1)^{n+t}}{\kappa - 1} \int_0^\infty v^{-\frac{1}{2}} dv \left(\frac{d}{dv}\right)^t \frac{(\frac{1}{2}\Delta)^{n-2t-2} \mathrm{R}^{n-t-1}}{\mathrm{L}(n-2t-2) \,.\, \mathrm{L}(n-t-1)} \frac{\phi}{\mathrm{Q}},$$

where Σ here implies summation from $n = 2t + 2$ to ∞.

To effect this summation we take the series for $\Delta^g \mathrm{R}^h$ given in Art. 21, where $g < h$. Putting $g = n - 2t - 2$, $h = n - t - 1$, we find

$$\frac{(-\frac{1}{2}\Delta)^g \mathrm{R}^h}{\mathrm{L}g \,.\, \mathrm{L}h} = \frac{\mathrm{R}^{t+1}(-\frac{1}{2}\mathrm{S})^g}{\mathrm{L}(t+1)\,\mathrm{L}(g)} + \frac{\mathrm{R}^{t+2}(-\frac{1}{2}\mathrm{S})^{g-2}\mathrm{T}}{\mathrm{L}(t+2)\,\mathrm{L}(g-2)\,\mathrm{L}(1)} \frac{1}{2^2} + \ldots$$

Using TAYLOR's theorem in the manner described in Art. 21, the result of performing this operation on ϕ is

$$\left\{\frac{\mathrm{R}^{t+1}}{\mathrm{L}(t+1)} + \frac{\mathrm{R}^{t+2}}{\mathrm{L}(t+2)} \frac{\mathrm{T}}{1} \cdot \frac{1}{2^2} + \frac{\mathrm{R}^{t+3}}{\mathrm{L}(t+3)} \frac{\mathrm{T}^2}{\mathrm{L}(2)} \frac{1}{2^4} + \ldots\right\} \phi\left(x - \frac{\lambda}{\alpha + v}, \&c.\right).$$

We now make the changes in the operators described in the article just referred to, and the expression for the potential becomes

$$\mathrm{V} = \frac{\pi(-1)^t}{\kappa - 1} \int_0^\infty v^{-\frac{1}{2}} dv \left(\frac{d}{dv}\right)^t \frac{1}{\mathrm{Q}} \mathrm{M}\phi\left(\frac{vx}{\alpha + v}, \frac{vy}{\beta + v}, \frac{vz}{\gamma + v}\right),$$

where

$$M = \frac{R^{t+1}}{L(t+1)} + \frac{R^{t+2}}{L(t+2)} \frac{T}{1} \frac{1}{2^2} + \ldots + \frac{R^{t+n}}{L(t+n)} \frac{T^n}{L(n)} \frac{1}{2^{2n}} + \ldots,$$

$$R = 1 - \frac{\alpha v x^2}{\alpha + v} - \frac{\beta v y^2}{\beta + v} - \frac{\gamma v z^2}{\gamma + v},$$

$$T = \frac{1}{v^2}\left\{(\alpha + v)\frac{d^2}{dx^2} + (\beta + v)\frac{d^2}{dy^2} + (\gamma + v)\frac{d^2}{dz^2}\right\}.$$

It is often more convenient to use the axes a, b, c of the ellipsoid instead of the squared reciprocals α, β, γ. Let us also suppose that the density at the point x, y, z, is $\psi(x/a, y/b, z/c)$. Putting $v = 1/u$, we have

$$R = 1 - \frac{x^2}{a^2 + u} - \frac{y^2}{b^2 + u} - \frac{z^2}{c^2 + u},$$

$$D = \frac{T}{u} = \frac{a^2 + u}{a^2}\frac{d^2}{dx^2} + \frac{b^2 + u}{b^2}\frac{d^2}{dy^2} + \frac{c^2 + u}{c^2}\frac{d^2}{dz^2},$$

$$Q_1^2 = (a^2 + u)(b^2 + u)(c^2 + u).$$

The potential at an internal point of the solid heterogeneous ellipsoid is then

$$V = \frac{\pi abc}{\kappa - 1}\int_0^\infty u^{-\frac{3}{2}}\, du \left(u^2 \frac{d}{du}\right)^t \frac{u^{\frac{3}{2}}}{Q_1} M\psi\left(\frac{ax}{a^2 + u}, \frac{by}{b^2 + u}, \frac{cz}{c^2 + u}\right),$$

$$M = \frac{R^{t+1}}{L(t+1)} + \frac{R^{t+2}}{L(t+2)} \frac{uD}{1} \frac{1}{2^2} + \ldots + \frac{R^{t+n+1}}{L(t+n+1)} \frac{u^n D^n}{L(n)} \frac{1}{2^{2n}} + \ldots,$$

and $t = \frac{1}{2}(2 - \kappa)$ where κ is the index of the law of force.

As an example, let the law be the inverse square, we then have $\kappa = 2$ and $t = 0$. The expression then becomes

$$V = \pi abc \int_0^\infty \frac{du}{Q_1} M\psi\left(\frac{ax}{a^2 + u}, \text{\&c.}\right)$$

$$M = R + \frac{R^2}{L(2)} \frac{uD}{1} \frac{1}{2^2} + \ldots + \frac{R^{n+1}}{L(n+1)} \frac{u^n D^n}{L(n)} \frac{1}{2^{2n}} + \ldots,$$

which agrees with that given by Mr. DYSON in the 'Quarterly Journal,' 1891.

31. In CAYLEY'S memoir on Prepotentials we find a simple and elegant formula for the potential of an ellipsoid in the form of a definite integral (see 'Phil. Trans.,' 1875, p. 745). Taking the notation of this paper, let V′ be the potential of a solid ellipsoid when the force varies as the inverse κ^{th} power of the distance at an *external* point (x, y, z). When the density ρ at an internal point is given by

$$\rho = \left(1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2}\right)^{n+t},$$

where a, b, c are the semi-axes of the bounding surface, it is shown that

$$(\kappa - 1)\,\mathrm{V}' = \frac{\{\Gamma(\frac{1}{2})\}^3\,\Gamma(1+t+n)}{\Gamma(\frac{3}{2}+t)\,\Gamma(1+n)}\,abc\int_{\epsilon^2}^{\infty}\frac{du}{u^{t+1}\mathrm{Q}_1}\cdot\left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right)^n,$$

where $t = \frac{1}{2}(\kappa - 4)$ and Q_1, ϵ^2 have the meaning given in Art. 31. It is necessary, also, that each of the three quantities $\frac{3}{2} + t$, $1 + n$, $1 + t + n$ should be positive.

When the ellipsoid is homogeneous $n + t = 0$, and the restrictions just mentioned require that the index κ of the force should lie between $-\frac{3}{2}$ and 4. When the index κ is greater than 4, the index $n + t$ of the density ρ cannot be less than $\frac{1}{2}(\kappa - 6)$.

Lejeune-Dirichlet (Liouville's 'Journal,' 1839, p. 165) gives a similar formula (also mentioned by Cayley) for the attraction of a solid homogeneous ellipsoid, but the index κ of the law of force is restricted to lie between 2 and 3.

In the 'Cambridge and Dublin Mathematical Journal,' vol. 3, 1843, Cayley has given the value of a multiple integral which, when the number of variables is reduced to three, is equivalent to the potential of a homogeneous ellipsoid. Taking as before the notation of this paper, and supposing the force to vary as the inverse κ^{th} power of the distance, the potential at an external point x, y, z becomes

$$\mathrm{V}' = \frac{\pi^{\frac{3}{2}}abc}{(\kappa - 1)\;2^{2t-2}\;\mathrm{L}\,(t)\;\Gamma\,(\frac{3}{2}+t)}\;\Delta^{t-1}\;\frac{1}{\lambda^2\mathrm{Q}\left\{\dfrac{x^2}{(\lambda+a^2)^2} + \dfrac{y^2}{(\lambda+b^2)^2} + \dfrac{z^2}{(\lambda+c^2)^2}\right\}},$$

where $t = \frac{1}{2}(\kappa - 4)$, $\quad \mathrm{Q}^2 = (\lambda + a^2)(\lambda + b^2)(\lambda + c^2)$,

$$\Delta = d^2/dx^2 + d^2/dy^2 + d^2/dz^2,$$

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

It is obvious that t must be an integer, so that κ must be an even integer and must be greater than 4.

As the subject of operation is not an integral rational function of x, y, z, this formula has the disadvantage that the algebraical work of using the operator is very long.

An elementary proof of this result will be given in Art. 37, by which it will appear that it is really a case of Jellet's theorem.

Another method of finding the potential of a heterogeneous ellipsoid has been given by Dr. Ferrers in the 'Quarterly Journal,' 1875. The method requires that the potential V of an ellipsoid, whose density is $\rho = (1 - \alpha\xi^2 - \beta\eta^2 - \gamma\zeta^2)^n$, should be known for all integer values of n. To deduce the potential V′, when the density at (ξ, η, ζ) is $\rho' = \xi^f\eta^g\zeta^h$ he finds a general expression for ρ' in a series of differential coefficients of ρ with descending powers of n. The value of V′ is then expressed in a corresponding series of differential coefficients of V. The law of force is supposed to be the inverse square, but the method applies for other laws of force, provided the fundamental expression for V is known.

32. When the attracted point is external, we may for all laws of force deduce the resolved attractions of a homogeneous ellipsoid by the use of IVORY's theorem, the attractions at an internal point having been found in Arts. 27, 28. But when we know the potential of a thin homœoid at an external point, we may also express as an integral the potential of a solid ellipsoid in which the density ρ at any point (ξ, η, ζ) is given by

$$\rho = f(\xi^2/a^2 + \eta^2/b^2 + \zeta^2/c^2).$$

When the force varies as the inverse square, this problem has in effect been solved by POISSON. The potential at an external point (x', y', z') is in that case

$$V' = \pi abc \int_0^\infty \{f_1(1) - f_1(m^2)\} \frac{du}{Q_1}$$

where $f_1(m)$ is the integral of $f(m)$, and

$$m^2 = \frac{x'^2}{a^2+u} + \frac{y'^2}{b^2+u} + \frac{z'^2}{c^2+u}, \quad Q_1{}^2 = (a^2+u)(b^2+u)(c^2+u).$$

Let us turn our attention to the corresponding integral when the law of force is the inverse κ^{th} power, where κ is greater than the square.

The potential of a thin homogeneous homœoid bounded by ellipsoids, whose major axes are a and $a(1+\mu)$ at an external point, has been found in a finite series in Art. 8. Let us use this to find the potential of a thin shell bounded by ellipsoids, whose major axes are ma and $(m + dm)\,a$. The density of the shell is $f(m^2)$, and the integral of its potential between the limits $m = 0$ to 1 is the potential of the heterogeneous solid ellipsoid.

Referring then to the formula in Art. 8, we unite dm/m for μ; ma, mb, mc, for a, b, c; and $m^2a^2 + \lambda$, $m^2b^2 + \lambda$, $m^2c^2 + \lambda$ for a'^2, b'^2, c'^2. We also put $\lambda = m^2u$ to simplify our results.

Since the attracted point lies on the confocal a', b', c', we have

$$\frac{x'^2}{a^2+u} + \frac{y'^2}{b^2+u} + \frac{z'^2}{c^2+u} = m^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The potential of the solid ellipsoid is therefore

$$V' = \frac{2\pi}{(\kappa-1)(\kappa-3)}\int_0^1 \frac{abc}{Q_1}\left(\frac{2}{Hu}\right)^{\kappa-3} f(m^2)\,mdm\left\{1+\frac{1}{2^2}\frac{HuD}{\kappa-4}+\ldots\right\}K^{\frac{1}{2}(\kappa-4)},$$

where

$$Q_1^{\,2} = (a^2+u)(b^2+u)(c^2+u),$$

$$H = \frac{x'^2}{(a^2+u)^2}+\frac{y'^2}{(b^2+u)^2}+\frac{z'^2}{(c^2+u)^2},$$

$$K = \frac{a^2x'^2}{(a^2+u)^3}+\frac{b^2y'^2}{(b^2+u)^3}+\frac{c^2z'^2}{(c^2+u)^3},$$

$$D = \frac{(a^2+u)^2}{a^2}\frac{d^2}{dx'^2}+\frac{(b^2+u)^2}{b^2}\frac{d^2}{dy'^2}+\frac{(c^2+u)^2}{c^2}\frac{d^2}{dz'^2},$$

the quantities represented by H, K, D being derived from E′, P′, Δ′ respectively by the changes described above.

Since by differentiating (1) we have

$$-2m\frac{dm}{du} = \frac{x'^2}{(a^2+u)^2}+\frac{y'^2}{(b^2+u)^2}+\frac{z'^2}{(c^2+u)^2} = H \quad . \quad . \quad . \quad . \quad . \quad (2),$$

we may conveniently change the independent variable from m to u, the limits being $u=\infty$ to $u=\epsilon^2$ where $a^2+\epsilon^2$, $b^2+\epsilon^2$, $c^2+\epsilon^2$ are the squared semi-axes of the ellipsoid passing through the attracted point, and confocal with the external surface of the given ellipsoidal body. We thus have

$$V' = \frac{2\pi}{(\kappa-1)(\kappa-3)}\cdot\int_\infty^{\epsilon^2}\frac{-abc}{Q_1}\frac{f(m^2)\,du}{u}\left(\frac{2}{Hu}\right)^{\kappa-4}\left\{1+\frac{1}{2^2}\frac{HuD}{\kappa-4}+\ldots\right\}K^{\frac{1}{2}(\kappa-4)},$$

where m^2 has the value given by (1).

This formula expresses the potential of a solid ellipsoid, whose strata of equal density are ellipsoids similar to the bounding surface, the law of force being the inverse κ^{th} power of the distance where κ is not less than the fourth. The series has $\frac{1}{2}(\kappa-2)$ terms.

We shall now apply this equation to the special cases in which the force varies as the inverse fourth, sixth, eighth and tenth powers of the distance.

33. *Suppose the law of force to be the inverse fourth,* then $\kappa=4$, and the series reduces to its first term. The potential of the ellipsoid at an external point is therefore

$$V' = \frac{2\pi}{3}\int_\infty^{\epsilon^2}\frac{-abc}{(a^2+u)^{\frac{1}{2}}(b^2+u)^{\frac{1}{2}}(c^2+u)^{\frac{1}{2}}}\frac{f(m^2)\,du}{u}.$$

If the density is uniform, $f(m^2) = 1$, and we fall back on the potential already obtained in Art. 5.

Let us, however, consider what forms of $f(m^2)$ will enable us to effect the integrations. Solving (1), Art. 32, we have

$$m^2 = \frac{Au^2 + 2Bu + C}{(a^2 + u)(b^2 + u)(c^2 + u)}$$

where

$$A = x'^2 + y'^2 + z'^2$$
$$2B = (x'^2 + y'^2 + z'^2)(a^2 + b^2 + c^2) - a^2x'^2 - b^2y'^2 - c^2z'^2$$
$$C = a^2b^2c^2 (x'^2/a^2 + y'^2/b^2 + z'^2/c^2).$$

If we put $f(m^2) = M/m$, where M is some constant, the potential reduces to the form

$$V' = \frac{2\pi}{3}\int_{\infty}^{\epsilon^2} \frac{-abcM}{(Au^2 + 2Bu + C)^{\frac{1}{2}}}\frac{du}{u}.$$

Writing $u = 1/v$ this takes the simple form

$$V' = \frac{2\pi}{3} abc \int \frac{M\,dv}{(A + 2Bv + Cv^2)^{\frac{1}{2}}} \quad v = 0 \text{ to } 1/\epsilon^2$$

where ϵ^2 is given by

$$\frac{x'^2}{a^2 + \epsilon^2} + \frac{y'^2}{b^2 + \epsilon^2} + \frac{z'^2}{c^2 + \epsilon^2} = 1.$$

This integral can be easily evaluated, but as the result is rather long it seems unnecessary to reproduce it here.

If the radius vector of the ellipsoid through any internal point R, whose coordinates are x, y, z, meet the surface in Q, we know that

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \left(\frac{OR}{OQ}\right)^2$$

where O is the centre. It follows that when the law of force is the inverse fourth power and the density at any point R varies as OQ/OR, the potential at any external point can be found in finite terms free from all signs of integration.

This theorem can be generalized. For if we take the density to be

$$\rho = F(m^2)/m$$

where F is a rational fraction, we arrive at an integral which can be evaluated by the use of partial fractions. The potential of the ellipsoid can therefore be found in this case also.

There is an analogous case where the force varies as the inverse square of the distance, and the density of the ellipsoid follows the law $\rho = F(m^2)/m$. This is discussed by POISSON, in the '*Connaissance des Tems*' for 1837. After alluding to a statement of JACOBI, that the attraction of certain heterogeneous ellipsoids can be expressed in finite terms, he shows how this result follows from the formulæ for the attraction of an ellipsoid which he had previously given in *volume* 13 of the '*Mémoires de l'Académie des Sciences.*' POISSON remarks that the simple case is not the homogeneous ellipsoid, but that in which the density varies inversely as m.

34. Another interesting case is that in which

$$\rho = f(m^2) = \frac{M}{(1 - m^2)^{\frac{1}{2}}},$$

where, as before, M is any constant. Substituting, we have

$$V' = \frac{2\pi}{3} abc \int_\infty^{\epsilon^2} \frac{-M\,du}{uQ_1}\left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right)^{-\frac{1}{2}},$$

where

$$Q_1^{\,2} = (a^2+u)(b^2+u)(c^2+u),$$

and ϵ^2 is the positive root of

$$\frac{x'^2}{a^2+\mu} + \frac{y'^2}{b^2+\mu} + \frac{z'^2}{c^2+\mu} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A).$$

If μ, μ', μ'' are the three roots of equation (A), we may obviously write the potential in the form

$$V' = \frac{2\pi}{3} abc \int_\infty^{\mu} \frac{du}{u} \frac{-M}{(u-\mu)^{\frac{1}{2}}(u-\mu')^{\frac{1}{2}}(u-\mu'')^{\frac{1}{2}}},$$

where $\mu = \epsilon^2$ has been chosen to be the positive root.

We may generalize this result by taking

$$\rho = M(1 - m^2)^{n+\frac{1}{2}},$$

the potential of the ellipsoid is then

$$V' = \frac{2\pi}{3} abc. \int_\infty^{\epsilon^2} \frac{-M\,du}{uQ_1}\left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right)^{n+\frac{1}{2}}.$$

The first of these two cases has some importance, for when the least axis (say c) is put zero, the ellipsoid reduces to a homogeneous elliptic disc. The surface density is then $\sigma = Mc\pi$. The expression for the potential, the law of force being the inverse fourth, is then

$$V' = \tfrac{2}{3} ab\sigma \int_\infty^{\mu} \frac{-du}{uQ_1}\left\{1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{u}\right\}^{-\frac{1}{2}}.$$

We may verify this result by comparing it with that given by CAYLEY in the 'Proceedings of the Mathematical Society,' 1875, for the potential of a homogeneous elliptic disc where the force varies as the inverse square of the distance. His result is

$$V = 2ab\sigma \int_\infty^\mu \frac{-du}{Q} \left\{1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{u}\right\}^{\frac{1}{2}}.$$

If V_κ be the potential of a disc when the law of force is the inverse κ^{th} power, we have

$$V_{\kappa+2} = -\frac{1}{\kappa+1}\frac{dV_\kappa}{zdz}.$$

This theorem is proved by ROBERTS in the 'Quarterly Journal,' 1881. Effecting the differentiation indicated, the two values of the potential are seen to agree, though the modes of demonstration are totally different. Dr. FERRERS has given an extension of CAYLEY's theorem on the potential of a disc, which, when differentiated, also agrees with the results given above, see the 'Quarterly Journal,' 1875.

35. Let us next consider the potential at an external point when *the force varies as the inverse sixth power*.

Here $\kappa = 6$ and the series in Art 32 is reduced to the two first terms. We have

$$V' = \frac{2\pi}{5.3}\int_\infty^{\epsilon^2} \frac{-abc}{Q_1}\frac{f(m^2)du}{u}\left(\frac{2}{Hu}\right)^2\left\{K + \frac{Hu}{4.2}DK\right\},$$

where H, K, D have the meanings given in the article referred to. We notice that, if p' be the perpendicular from the centre on the tangent plane to the confocal at the attracted point,

$$H = \frac{1}{p'^2}, \qquad K = \frac{1}{2u}\frac{d}{du}(Hu^2)$$

$$DK = 2\left\{\frac{1}{a^2+u} + \frac{1}{b^2+u} + \frac{1}{c^2+u}\right\} = \frac{4}{Q_1}\frac{dQ_1}{du}$$

$$Q_1^2 = (a^2+u)(b^2+u)(c^2+u).$$

Substituting in the expression for V' we have

$$V' = \frac{4\pi}{5.3}\int_\infty^{\epsilon^2} abc\left\{-\frac{2p'^2}{u^3Q_1} + \frac{2p'}{u^2Q_1} - \frac{p'^2}{u^2{Q_1}^2}\frac{dQ_1}{du}\right\}du\,.f(m^2)$$

$$= \frac{4\pi}{5.3}abc\int_\infty^{\epsilon^2} f(m^2)\frac{d}{du}\left(\frac{p'^2}{u^2Q_1}\right)du,$$

where ϵ^2 is the positive root of

$$\frac{x'^2}{a^2+\epsilon^2} + \frac{y'^2}{b^2+\epsilon^2} + \frac{z'^2}{c^2+\epsilon^2} = 1.$$

If the ellipsoid is homogeneous, we have $f(m^2) = 1$, and the potential takes the simple form

$$V' = \frac{4\pi}{5.3} \frac{abc\, p'^2}{\epsilon^4 (a^2+\epsilon^2)^{\frac{1}{2}} (b^2+\epsilon^2)^{\frac{1}{2}} (c^2+\epsilon^2)^{\frac{1}{2}}}.$$

It appears that for all points on the same confocal the potential varies as the square on the tangent plane.

36. When the density is not uniform, we have by an integration by parts

$$V' = \frac{4\pi}{5.3} abc \left\{ \frac{f(1)\, p'^2}{\epsilon^4 a'b'c'} + \int_\infty^{\epsilon^2} \frac{f'(m^2)\, du}{u^2 Q_1} \right\},$$

since $m^2 = 1$ when $u = \epsilon^2$, and by equation (2) of Art. 32 $dm^2/du = -H = -1/p'^2$. We have also written a', b', c' for the semi-axes of the confocal which passes through the attracted point.

When the density $f(m^2) = Mm$ where M is any constant, we have $f'(m^2) = M/2m$. We then find (see Art. 33)

$$V' = \frac{4\pi}{5.3} abc \frac{M}{2} \left\{ \frac{p'^2}{\epsilon^4 a'b'c'} + \int_\infty^{\epsilon^2} \frac{du}{u^2 (Au^2 + 2Bu + C)^{\frac{1}{2}}} \right\}.$$

The integration can be effected by writing $u = 1/v$.

37. The expression for the potential of a homogeneous ellipsoid, when the law of force is the inverse sixth power, having been found freed from all signs of integration, it follows by JELLETT's theorem that the potential for any even power greater than 6 can also be expressed in an integrated form (Art. 5). In fact, if V_κ be the potential of the ellipsoid when the force varies as the inverse κ^{th} power, we have when $t = \frac{1}{2}(\kappa - 4)$

$$V = \frac{8\pi\, abc}{(\kappa-1)\, L(\kappa-3)} \Delta^{t-1} \frac{p'^2}{\epsilon^4 (a^2+\epsilon^2)^{\frac{1}{2}} (b^2+\epsilon)^{\frac{1}{2}} (c^2+\epsilon^2)^{\frac{1}{2}}}.$$

Writing λ for ϵ^2 and remembering the expression for the perpendicular p' on the tangent plane to the confocal, this is easily seen to agree with CAYLEY'S formulæ quoted in Art. 31.

38. *When the force varies as the inverse eighth power*, $\kappa = 8$, and the series for the potential is reduced to three terms. We thus have (Art. 32)

$$V' = \frac{2\pi}{7.5} abc \int_\infty^{\epsilon^2} \frac{-f(m^2)\, du}{Q_1 u} \left(\frac{2}{Hu}\right)^4 \left\{ K^2 + \frac{Hu}{4.4} DK^2 + \frac{H^2u^2}{2^4.8.3} D^2K^2 \right\}.$$

Referring to the values of H, K already written down (Art. 32), we find

$$DK^2 = 8 \left\{ \frac{a^2 x'^2}{(a^2+u)^4} + \ldots \right\} + 4K \left\{ \frac{1}{a^2+u} + \ldots \right\}$$

which may be written shortly in the form

$$DK^2 = -\frac{8}{3}\frac{dK}{du} - 8\,KQ_1\frac{d}{du}\frac{1}{Q_1},$$

where the differentiations with regard to u are partial, the co-ordinates x', y', z' of the attracted point not being varied. Similarly we have

$$D^2K^2 = 32\,Q_1\frac{d^2}{du^2}\frac{1}{Q_1}.$$

Substituting these in the value of V′, we find, after some reductions,

$$V' = -\frac{8\pi}{7\,.\,5\,.\,3}\,abc\int_{\infty}^{\epsilon^2}(fm^2)\,du\,\frac{d}{du}\left\{p'^2\frac{d}{du}\left(\frac{p'^2}{u^3Q_1}\right)\right\}.$$

If the ellipsoid is homogeneous, the potential at an external point, when the law of force is the inverse eighth, is

$$V' = -\frac{8\pi}{7\,.\,5\,.\,3}\,abc\,.\,p'^2\frac{d}{du}\left(\frac{p'^2}{u^3Q_1}\right),$$

where $u = \epsilon^2$.

When written at length this becomes

$$V' = \frac{4\pi}{7\,.\,5\,.\,3}\,abc\,\frac{p'^4}{u^3Q_1}\left[\frac{6}{u} + \frac{1}{a^2+u} + \frac{1}{b^2+u} + \frac{1}{c^2+u} - 4p^2\left\{\frac{x^2}{(a^2+u)^3} + \frac{y^2}{(b^2+u)^3} + \frac{z^2}{(c^2+u)^3}\right\}\right].$$

When the density is not uniform we have, after twice integrating by parts,

$$V' = -\frac{8\pi}{7\,.\,5\,.\,3}\,abc\left[f(1)\,p'^2\frac{d}{du}\left(\frac{p'^2}{u^3Q_1}\right) + f'(1)\frac{p'^2}{u^3Q_1} + \int_{\infty}^{\epsilon^2}\frac{f''(m)}{u^3Q_1}\,du\right],$$

where $u = \epsilon^2$ in the integrated parts.

This result takes a very simple form when the density is given by $f(m^2) = m^2$. The unintegrated term is then zero, and the potential reduces to the two first terms.

39. When the law of force is the inverse tenth power of the distance the expression becomes more complicated. Putting

$$U = \frac{p'^2}{u^2}\left[\frac{u^2}{Q_1}\frac{d^2}{du^2}\left(\frac{p'}{u}\right)^4 + \left(3u^2\frac{d}{du}\frac{1}{Q_1} - \frac{4u}{Q_1}\right)\frac{d}{du}\left(\frac{p'}{u}\right)^4 + \left(2u^2\frac{d^2}{du^2}\frac{1}{Q_1} - 4u\frac{d}{du}\frac{1}{Q_1} + \frac{4}{Q_1}\right)\left(\frac{p'}{u}\right)^4\right]$$

where the differentiations with regard to u are partial, we find, after integrating by parts, that the potential of a homogeneous ellipsoid at an external point is

$$V' = \frac{8\pi}{9\,.\,7\,.\,5\,.\,3}\,abc\,U.$$

When the density is not uniform the potential is

$$V' = \frac{8\pi}{9.7.5.3} abc \left[f(1) U + \int_\infty^{\epsilon^2} \frac{U f'(m^2)}{p'^2} du \right].$$

40. When the index of the law of force is less than 4, the formula in Art. 32 becomes inconvenient because the index of K is negative. To find the potential of a solid ellipsoid at an external point, we must, in this case, integrate the expression given in Art. 26 for the potential of an elementary shell. The integration, however, presents great difficulties except in the case in which the law of force is the inverse square. The method of proceeding is then as follows:

To find the potential of a thin homœoid whose density is $x^f y^g z^h$ and axes (ma, mb, mc) and $((m + dm) a, \&c.)$ we use the formula of Art. 26. Writing dm/m for μ, ma, mb, mc for a, b, c, $m^2\omega$ for u, and $m^2\lambda$ for ϵ^2, we find that the potential of the shell is

$$\pi abc\, dm^2 \int_\lambda^\infty \Sigma \frac{(m^2 - 1 + R)^n \omega^n D^n}{L(n) L(n) 2^{2n}} \left(\frac{a^2 x}{a^2 + \omega}\right)^f \left(\frac{b^2 y}{b^2 + \omega}\right)^g \left(\frac{c^2 z}{c^2 + \omega}\right)^h \frac{d\omega}{Q_1},$$

where R, D have the meaning given to them in Art. 22, except that ω replaces u, and Σ implies summation from $n = 0$ onwards.

To deduce the potential of the heterogeneous ellipsoid we integrate this from $m^2 = 0$ to 1.

We notice that

$$m^2 = \frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda},$$

let μ be the value of λ when $m = 1$, so that $a^2 + \mu$, $b^2 + \mu$, $c^2 + \mu$ are the squared semi-axes of that confocal to the boundary of the solid ellipsoid which passes through the attracted point, viz., x, y, z. Then, as m^2 passes from 0 to 1, λ passes from ∞ to μ.

Since $R = 1 - m^2$ when $\omega = \lambda$, we have

$$\frac{d}{dm^2} \int_\lambda^\infty \frac{(m^2 - 1 + R)^{n+1}}{n+1} f(\omega)\, d\omega = 0 + \int_\lambda^\infty \frac{d}{dm^2} \frac{(m^2 - 1 + R)^{n+1}}{n+1} f(\omega)\, d\omega$$

$$= \int_\lambda^\infty (m^2 - 1 + R)^n f(\omega)\, d\omega.$$

We shall now integrate this equation from $m^2 = 0$ to 1, *i.e.*, $\lambda = \infty$ to μ. The left side becomes

$$\int_\mu^\infty \frac{R^{n+1}}{n+1} f(\omega)\, d\omega - \int_\infty^\infty \frac{(R-1)^{n+1}}{n+1} f(\omega)\, d\omega.$$

But R and $f(\omega)$ are both zero for all infinite values of ω, hence

$$\int_\mu^\infty \frac{R^{n+1}}{n+1} f(\omega)\, d\omega = \int^1 dm^2 \int_\lambda^\infty (m^2 - 1 + R)^n f(\omega)\, d\omega.$$

The potential of a solid ellipsoid whose density is $x^f y^g z^h$ at an external point is therefore

$$V' = \pi abc \int_\mu^\infty \Sigma \frac{R^{n+1}\omega^n D^n}{L(n+1)\,L(n)\,2^{2n}} \left(\frac{a^2x}{a^2+\omega}\right)^f \left(\frac{b^2y}{b^2+\omega}\right)^g \left(\frac{c^2z}{c^2+\omega}\right)^h \frac{d\omega}{Q_1}.$$

This agrees with Mr. DYSON'S result, 'Quarterly Journal,' 1891.

41. When the force varies as an odd power of the distance, the analysis is different from that given above. There are two cases, according as the inverse index κ is negative or positive and greater than unity.

The former case is the easier, and ultimately depends on integrals resembling those used in finding moments of inertia, but of a higher power. It will be convenient to take this case first.

The latter case is the more interesting as we cannot suppose that the force can vary as a direct power of the distance except within a limited space. The integrals which present themselves in this case are similar to those used in the ordinary theory when the force varies as the inverse square. On account of their complexity, it will be useful to begin this part of the discussion with the simpler case, in which the force varies as the inverse cube, and then to proceed to the more general law of force.

Though we have here distinguished the cases according as the index κ is negative or positive, this is only to obtain finite results. The expression obtained for the potential when κ is an odd negative integer, may be applied to the case in which κ is even or odd, positive or negative, but the expression then takes the form of an infinite series.

42. Before proceeding to this formula, it will be useful to notice two other methods of arriving at the same potential.

Mr. W. D. NIVEN has given in the 'Phil. Trans.' for 1879, a useful formula resembling the one used in Art. 43. Let U be any function of the coordinates (ξ, η, ζ) of a point on an ellipsoid. Taking the integrations over the whole surface, we have

$$\iint Up\,dS = 4\pi abc\left(1 + \frac{\nabla}{L(3)} + \frac{\nabla^2}{L(5)} + \ldots\right)U$$

where

$$\nabla = a^2\,d^2/d\xi^2 + b^2\,d^2/d\eta^2 + c^2\,d^2/d\zeta^2,$$

and (ξ, η, ζ) are to be put zero after the differentiations are performed. There is a similar expression when the integration extends throughout the volume of the ellipsoid, and the strata of equal density are similar ellipsoids. By giving U different forms, he obtains the values of some useful integrals.

To apply this to find the potential of an ellipsoidal shell at a point (x, y, z) for the law of the inverse square, he puts

$$U = \{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^{\frac{1}{2}}.$$

He notices that the differentiations with regard to (ξ, η, ζ) can be changed into differentiations with regard to (x, y, z), and that, therefore, ξ, η, ζ may be put equal to zero before instead of after differentiation.

Dr. HOBSON has given another method in the 'Proceedings of the London Mathematical Society,' vol. 24, 1893. He finds the expansions in series of the potentials of ellipsoidal shells, solid ellipsoids and discs of variable density, the law of force being any given function of the distance. Taking the case of a homœoid, he first expands the surface density ρ in a series of spherical harmonics, say

$$\rho/p = \Sigma Y_n (x/a, y/b, z/c)$$

where p is the perpendicular on the tangent plane. The potential due to a surface distribution on the ellipsoid represented by any one of these terms, when the law of force is $-\phi'(R)$, is then given by the equation

$$\iint Y_n \phi(R)\, p\, d\sigma$$
$$= 4\pi abc(-1)^n \frac{2^n L(n)}{L(2n+1)} \left\{1 + \frac{\Delta}{2(2n+3)} + \&c.\right\} Y_n\left(a\frac{d}{dx}, b\frac{d}{dy}, c\frac{d}{dz}\right)\phi(r)$$

where $r^2 = x^2 + y^2 + z^2$, R is the distance of the attracted point from the element $p\, d\sigma$, and

$$\Delta = a^2 d^2/dx^2 + b^2 d^2/dy^2 + c^2 d^2/dz^2.$$

This series terminates when $\phi(R)$ is an integral rational function of R, and in other cases the potential is given by an infinite series.

43. *To find the potential of a thin homogeneous homœoid at an internal or external point* P *when the force varies as the inverse* κ^{th} *power, where* κ *is an odd negative integer.*

Let x, y, z be the coordinates of the attracted point P, ξ, η, ζ those of any elementary mass $\mu p\, d\sigma$ of the homœoid. Writing $1 - \kappa = 2t$ to avoid negative quantities, we see that the potential of the ellipsoidal shell is

$$V = \frac{\mu}{-2t}\iint p\, d\sigma\, [(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2]^t.$$

Let A, B, C, as before, be d/dx, d/dy, d/dz; then the potential is

$$V = \frac{\mu}{-2t}\iint p\, d\sigma e^{-A\xi - B\eta - C\zeta}(x^2 + y^2 + z^2)^t.$$

We shall now expand the exponential and integrate the several terms over the whole surface of the ellipsoid, x, y, z being constant. The terms of the expansion with odd exponents need not be considered, as after integration they are clearly zero. The integral for any even power is best obtained from the corresponding integral for a sphere by the method of projections. We have

$$\iint p\, d\sigma \frac{(\mathrm{A}\xi + \mathrm{B}\eta + \mathrm{C}\zeta)^{2s}}{\mathrm{L}(2s)} = 4\pi abc \frac{(a^2\mathrm{A}^2 + b^2\mathrm{B}^2 + c^2\mathrm{C}^2)^s}{\mathrm{L}(2s+1)},$$

where s is a positive integer.

Let $\Delta = a^2\, d^2/dx^2 + b^2\, d^2/dy^2 + c^2\, d^2/dz^2$.

Let the mass $4\pi\mu abc = \mathrm{M}$. The potential then takes the simple form

$$\mathrm{V} = -\frac{\mathrm{M}}{2t}\left\{1 + \frac{\Delta}{\mathrm{L}(3)} + \frac{\Delta^2}{\mathrm{L}(5)} + \ldots + \frac{\Delta^s}{\mathrm{L}(25+1)} + \ldots\right\}(x^2 + y^2 + z^2)^t.$$

Here $2t = 1 - \kappa$, so that the *series is finite* if the index of the force is an odd negative integer. The expansion is, however, clearly true whether t is integral or not, provided only the series is convergent.

44. *To find the potential of a heterogeneous homœoid when κ is an odd negative integer.*

Let the law of density be $\rho = \xi^f\eta^g\zeta^h$. Following the same reasoning as before, we have

$$\mathrm{V} = -\frac{\mu}{2t}\iint p d\sigma e^{-\mathrm{A}\xi - \mathrm{B}\eta - \mathrm{C}\zeta}(x^2 + y^2 + z^2)^t\, \xi^f\eta^g\zeta^h.$$

This is the same as

$$\mathrm{V} = -\frac{\mu}{2t}\iint p d\sigma \left(-\frac{d}{d\mathrm{A}}\right)^f \left(-\frac{d}{d\mathrm{B}}\right)^g \left(-\frac{d}{d\mathrm{C}}\right)^h e^{-\mathrm{A}\xi - \mathrm{B}\eta - \mathrm{C}\zeta}(x^2 + y^2 + z^2)^t.$$

Making the same expansion, we have

$$\mathrm{V} = -\frac{\mu}{2t}\left(-\frac{d}{d\mathrm{A}}\right)^f \left(-\frac{d}{d\mathrm{B}}\right)^g \left(-\frac{d}{d\mathrm{C}}\right)^h 4\pi abc\, \Sigma \frac{(\mathrm{A}^2a^2 + \mathrm{B}^2b^2 + \mathrm{C}^2c^2)^s}{\mathrm{L}(2s+1)} r^{2t}.$$

When the law of density is $\rho = \phi(\xi, \eta, \zeta)$, we have

$$\mathrm{V} = -\frac{4\pi\mu abc}{2t}\phi\left(-\frac{d}{d\mathrm{A}}, -\frac{d}{d\mathrm{B}}, -\frac{d}{d\mathrm{C}}\right)\left\{1 + \frac{\mathrm{A}^2a^2 + \mathrm{B}^2b^2 + \mathrm{C}^2c^2}{\mathrm{L}(3)} + \ldots\right\} r^{2t},$$

where $r^2 = x^2 + y^2 + z^2$, A, B, C stand for d/dx, d/dy, d/dz, and $2t = 1 - \kappa$. The series is finite when κ is an odd negative integer.

We may also deduce this expression from the series given in NIVEN's memoir.

Putting

$$U = \xi^f \eta^g \zeta^h \psi \{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\},$$

the general term of his expansion is

$$\frac{4\pi abc}{L(2i+1)} \left(a^2 \frac{d^2}{d\xi^2} + b^2 \frac{d^2}{d\eta^2} + c^2 \frac{d^2}{d\zeta^2}\right)^i U.$$

Let $\delta = d/d\xi$ and $A = d/dx$. Using the generalized theorem of LEIBNITZ for the differential coefficient of a product, and putting $\xi = 0$, after differentiation we have

$$\phi(\delta)\, \xi^f \psi \{(\xi - x)^2 + \ldots\} = \left(\frac{d}{d\delta}\right)^f \phi(\delta) \,.\, \psi\{(\xi - x)^2 + \ldots\} = \left(-\frac{d}{dA}\right)^f \phi(-A)\psi(x^2 + \ldots),$$

the last term being obtained by changing the subject of differentiation from ξ to x. Treating η^g and ζ^h in the same way, we have

$$\phi(\nabla)\, U = (-d/dA)^f (-d/dB)^g (-d/dC)^h \phi(A^2a^2 + B^2b^2 + C^2c^2)\, \psi(x^2 + y^2 + z^2).$$

When we replace $\phi(\nabla)$ by ∇^i and ψ by a t^{th} power, this becomes virtually identical with the general term of the expansion found above for V.

45. As an example, let the law of force be the direct cube of the distance, and let the density be $\rho = \xi$. We have

$$V = \frac{4\pi\mu abc}{4} \frac{d}{dA} \left\{ r^4 + \frac{A^2a^2 + \ldots}{L(3)} r^4 + \frac{(A^2a^2 + \ldots)^2}{L(5)} r^4 \right\}$$

$$= \pi\mu abc \left\{ \frac{2a^2}{L(3)} Ar^4 + \frac{4a^2}{L(5)} (A^2a^2 + B^2b^2 + C^2c^2)\, Ar^4 \right\}.$$

Remembering the meaning of A, B, C, this is clearly

$$V = \frac{4\pi\mu}{3 \,.\, 5} a^3bc \,.\, x\, (5r^2 + 3a^2 + b^2 + c^2).$$

The potential for a homogeneous homœoid for the same law of force is

$$V = -\frac{\pi\mu}{3} abc\, \{3r^4 + 4(a^2x^2 + b^2y^2 + c^2z^2) + 2(a^2 + b^2 + c^2)\, r^2\},$$

a constant term being omitted.

46. We may use the same method *to find the potential of a solid ellipsoid when κ is any odd negative integer.*

Proceeding as before, we have

$$V = -\frac{1}{2t}\int dv\,\{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^t = -\frac{1}{2t}\int dv\, e^{-A\xi - B\eta - C\zeta}\,(x^2 + y^2 + z^2)^t$$

where dv is an element of volume. Either by DIRICHLET'S theorem, or by projections from a sphere, we have

$$\int dv\,\frac{(A\xi + B\eta + C\zeta)^{2s}}{L\,(2s)} = \frac{4\pi abc}{2s+3}\,\frac{(A^2a^2 + B^2b^2 + C^2c^2)^s}{L\,(2s+1)}.$$

It immediately follows that for a *homogeneous solid ellipsoid* the potential is

$$V = -\frac{4\pi abc}{2t}\cdot\left\{\tfrac{1}{3} + \frac{A^2a^2 + \ldots}{L\,(3)\,.\,5} + \frac{(A^2a^2 + \ldots)^2}{L\,(5)\,.\,7} + \ldots\right\}(x^2 + y^2 + z^2)^t$$

where A, B, C stand for d/dx, d/dy, d/dz as before.

For a *heterogeneous ellipsoid* whose density is $\rho = \phi\,(\xi, \eta, \zeta)$, we have

$$V = -\frac{4\pi abc}{2t}\,\phi\left(-\frac{d}{dA}, -\frac{d}{dB}, -\frac{d}{dC}\right)\left\{\tfrac{1}{3} + \frac{A^2a^2 + \ldots}{L\,(3)\,.\,5} + \ldots\right\}(x^2 + y^2 + z^2)^t.$$

Both these expressions are finite where t is positive, *i.e.*, the index of the power is a negative odd integer.

47. *This method may be applied to other bodies besides the ellipsoid*, provided only the value of the fundamental integral

$$I = \int dv\,(A\xi + B\eta + C\zeta)^s$$

is known for that body. There are many bodies for which this integral is known, such as triangles, quadrilaterals, tetrahedra, and so on (see a paper by the author in the 'Quarterly Journal,' 1886). As an example, consider the tetrahedron, let (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), &c., be the coordinates of the four corners, let $\alpha = A\xi_1 + B\eta_1 + C\zeta_1$, $\beta = A\xi_2 + B\eta_2 + C\zeta_2$, &c., so that $\alpha, \beta, \gamma, \delta$ are proportional to the distances of the four corners from the plane of reference. The fundamental integral I is then equal to the product of the volume of the tetrahedron by the arithmetic mean of the homogeneous products of $\alpha, \beta, \gamma, \delta$ of the s^{th} degree.

By choosing a proper origin and axes the expressions $\alpha, \beta, \gamma, \delta$ may be simplified to suit any particular case. Using this expression for the fundamental integral the potential of a tetrahedron for any direct odd law of force can be deduced from the differential coefficients of r^{2t}.

48. When the index κ is not very large, we can often write down the potential

by using the theory of equimomental points. We have already seen that the potential of any body is

$$V = -\frac{\mu}{2t}\int dv\,\{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2\}^t.$$

Expanding the t^{th} power, we see that the highest powers of (ξ, η, ζ) which occur are in the term independent of x, y, z. These only give a constant term to the potential. Neglecting these terms we see that if two bodies are such that $\int \xi^f \eta^g \zeta^h\, dv$ is the same for both, when $f + g + h$ is $=$ or $< 2t - 1$, these bodies have equal potentials at all points. Now $2t - 1$ is the index of the force taken positively. Hence, if two bodies are equimomental for all powers up to the κ^{th}, they are also equipotential, except for a constant.

There are many bodies which may be replaced by equimomental points. For example, we can show that, up to cubes inclusive, a solid ellipsoid is equimomental to six particles placed at the extremities of any set of conjugate diameters, each equal to one-tenth of the mass of the ellipsoid together with a seventh particle placed at the centre of gravity equal to two-fifths of the mass of the ellipsoid. Placing the six particles at the extremities of the axes, we can write down the potential of the ellipsoid when the force varies as the cube of the distance. We clearly have

$$-4V = \tfrac{1}{10}M\,\{x^2 + y^2 + (z - c)^2\}^2 + \ldots + \tfrac{2}{5}M\,(x^2 + y^2 + z^2)^2,$$

therefore, omitting a constant,

$$V = -\tfrac{1}{20}M\,\{5r^4 + 2\,(a^2 + b^2 + c^2)\,r^2 + 4\,(a^2x^2 + b^2y^2 + c^2z^2)\}.$$

In the same way a tetrahedron, up to cubes inclusive, is equimomental to eight points. We collect nine-fortieths of the volume at the centre of gravity of each face and one-fortieth at each corner. The potential of the tetrahedron when the force varies as the direct cube of the distance can therefore be immediately written down. Let the centre of gravity G of the tetrahedron be the origin, and let the axis of z pass through the attracted point P. Then $GP = z$, and (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2), &c., are the coordinates of the four corners. The potential is easily seen to be

$$V = -\frac{M}{4}\left\{z^4 + \frac{z^2}{10}\,\Sigma\,(\xi^2 + \eta^2 + 3\zeta^2) - \frac{z}{15}\,\Sigma\,(\xi^2 + \eta^2 + \zeta^2)\zeta\right\}.$$

49. *To find the potential of a thin homœoid at an internal point P when the law of attraction is the inverse cube.*

Referring to Art. 3, the potential V for an odd inverse κ^{th} power is

$$V = \frac{\mu}{\kappa - 1}\iint \frac{r_1^{\,3-\kappa} + r_2^{\,3-\kappa}}{r_1 - r_2}\,D^2\,d\omega.$$

When $\kappa = 3$, this becomes

$$V_3 = \frac{\mu}{2}\iint \frac{2D^2}{r_1 - r_2}\,d\omega.$$

Taking the standard quadratic (3) of Art. 1, we have

$$\frac{r_1 - r_2}{2D^2} = \left\{F^2 + \frac{E}{D^2}\right\}^{\frac{1}{2}} = \{(\alpha lx + \beta my + \gamma nz)^2 + E\,(\alpha l^2 + \beta m + \gamma n^2)\}^{\frac{1}{2}}.$$

The potential required is, therefore,

$$V = \tfrac{1}{2}\,\mu\iint r'\,d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where r' is the radius vector of the auxiliary ellipsoid

$$(\alpha x X + \beta y Y + \gamma z Z)^2 + E\,(\alpha X^2 + \beta Y^2 + \gamma Z^2) = 1 \quad . \quad . \quad . \quad . \quad (2),$$

which may also be written in the form

$$(1 - \beta y^2 - \gamma z^2)\,\alpha X^2 + \ldots + 2\beta\gamma\,yz\,YZ + \ldots = 1 \quad . \quad . \quad . \quad . \quad (3).$$

To find the integral we require the following lemma :—

Lemma. *Let r' be the radius vector of an ellipsoid whose semi-axes are a' b', c'; r'' the radius vector of the polar reciprocal with regard to a concentric sphere of radius m. Then*

$$\iint r'\,d\omega = \frac{a'b'c'}{m^4}\iint r''^2\,d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4).$$

To prove this, we notice that

$$\iint r'\,d\omega = \iiint dr'\,d\omega = \iiint \frac{dx\,dy\,dz}{x^2 + y^2 + z^2},$$

the integration extending throughout the volume of the ellipsoid. Writing $x = a'\xi$, $y = b'\eta$, $z = c'\zeta$, and integrating throughout the volume of a sphere of radius unity, this becomes

$$\iiint \frac{a'b'c'\,d\xi\,d\eta\,d\zeta}{a'^2\xi^2 + b'^2\eta^2 + c'^2\zeta^2} = a'b'c'\iiint \frac{dr\,d\omega}{c'^2\cos^2\theta + \sin^2\theta\,(b'^2\cos^2\phi + a'^2\sin^2\phi)}$$

by changing to polar coordinates. After integrating with regard to r from 0 to 1, it becomes obvious that the double intregral remaining is the value of $\iint r''^2\,d\omega$ for an ellipsoidal surface whose semi-axes are $1/a'$, $1/b'$, $1/c'$. Introducing the m to preserve the dimensions, the theorem follows at once.

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50. It follows from this lemma that the potential of the homœoid is

$$V_3 = \frac{\mu a'b'c'}{2m^4} \iint r''^2 \, d\omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5),$$

where a', b', c' are the semi-axes of the auxiliary ellipsoid (3).

This double integral occurs in the ordinary expression for the potential at the centre of a homœoid when the law of force is the inverse square, and its axes are m^2/a', m^2/b', m^2/c'. Taking its value from that investigation, we have

$$\iint r''^2 \, d\omega = \frac{2\pi m^4}{a'b'c'} \int_0^\infty \frac{du}{Q'}$$

where

$$Q'^2 = (\alpha' + u)(\beta' + u)(\gamma' + u) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6),$$

and α', β', γ' are the squared reciprocals of the semi-axes of the auxiliary ellipsoid. We, therefore, have

$$V_3 = \pi\mu \int_0^\infty \frac{du}{Q'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7).$$

The axes of the auxiliary ellipsoid are given by its discriminating cubic. Let R be the leading letter of the cubic, so that the three values of R are α', β', γ'. As we want the product represented above by Q'^2, we change the cubic into another having S for its leading letter by writing $R + u = S$. The last term of this second cubic is the product required. In this way we find

$$Q'^2 = \begin{vmatrix} (1-\beta y^2-\gamma z^2)\,\alpha+u, & \alpha\beta xy, & \alpha\gamma xz \\ \alpha\beta xy, & (1-\gamma z^2-\alpha x^2)\,\beta+u, & \beta\gamma yz \\ \alpha\gamma xz, & \beta\gamma yz, & (1-\alpha x^2-\beta y^2)\,\gamma+u \end{vmatrix} \quad . \quad . \quad (8).$$

Substituting either this determinant or the product (6) in (7), we have an expression for V_3.

If M be the mass of the homœoid, the potential at an internal point when the law of force is the inverse cube is given by

$$V_3 = \frac{M}{4abc} \int_0^\infty \frac{du}{Q'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9).$$

We may put this theorem into a geometrical form by remembering that in the equation (5) the double integral occurs in finding the potential of a homœoid when the law of force is the inverse square. We have, therefore, the following theorem.

Let the potential of a thin homœoid of mass M, and semi-axes a, b, c, be V_3, when the force varies as the inverse cube. Let the potential of a second thin homœoid of mass M″ at an internal point be V_2'', when the force varies as the inverse square. If the bounding surface of the second homœoid be the polar reciprocal with regard to a sphere of radius m of the auxiliary ellipsoid of the first, then

$$\frac{V_3}{M} = \frac{V_2''}{M''}\,\frac{m^2}{2abc}.$$

The potential of the homœoid at an external point P′ is the same as that of a thin confocal homœoid passing through P′ of equal volume and density at the internal corresponding point P.

51. To discover the geometrical relation of the auxiliary ellipsoid to the homœoid whose potential at P is required, we examine the form (2) of Art. 49. Let OP produced cut the homœoid in C, and let OA, OB, OC, be a set of conjugate diameters. Then, if these lines are taken as a new set of axes, the equations of the homœoid and its auxiliary ellipsoid respectively become

$$\frac{x'^2}{OA^2} + \frac{y'^2}{OB^2} + \frac{z'^2}{OC^2} = 1,$$

$$\frac{z'^2}{OP^2} + \left(\frac{x'^2}{OA^2} + \frac{y'^2}{OB^2} + \frac{z'^2}{OC^2}\right)E = 1.$$

The two are thus referred to a common set of conjugate diameters. If we take A′, B′, C′, in OA, OB, OC, respectively, so that

$$\left(\frac{OA}{OA'}\right)^2 = \left(\frac{OB}{OB'}\right)^2 = 1 - \frac{OP^2}{OC^2}, \quad \left(\frac{OC}{OC'}\right)^2 = 1 + \left(\frac{OC}{OP}\right)^2 - \left(\frac{OP}{OC}\right)^2,$$

the ellipsoid having OA′, OB′, OC′ for conjugate diameters is the auxiliary ellipsoid.

52. *To find the potential of a homogeneous thin homœoid at an internal point* P, *when the law of attraction is any inverse odd power greater than* 3.

Putting $u = 1/r$, the required potential, by Art. 3, becomes

$$V = \frac{\mu}{\kappa - 1}\iint \frac{u_1^{\kappa-3} + u_2^{\kappa-3}}{u_1 - u_2}\,\frac{d\omega}{E}.$$

Referring to the standard quadratic (3) of Art. 1, and putting $p = 2F/E$, $q = 1/ED^2$, we have by the theorem already quoted in Art. 8,

$$u_1^n + u_2^n = p^n + np^{n-2}q + \ldots + n\frac{L(n-f-1)}{L(f)\,L(n-2f)}\,p^{n-2f}q^f + \ldots$$

This series stops at the first negative power of p and it is necessary that n should be greater than zero. If $n = 0$, the left hand side must be halved. In our case we put $n = \kappa - 3$, and since κ is odd the series has $\frac{1}{2}n + 1$, *i.e.*, $\frac{1}{2}(\kappa - 1)$ terms.

As in Art. 49, we have

$$u_1 - u_2 = \frac{2}{\mathrm{E}}\left(\mathrm{F}^2 + \frac{\mathrm{E}}{\mathrm{D}^2}\right)^{\frac{1}{2}} = \frac{2}{\mathrm{E}}\frac{1}{r'},$$

where r' is the radius vector of the same auxiliary ellipsoid as that used for the inverse cube. In this way we find

$$\mathrm{V} = \frac{\mu}{2(\kappa - 1)}\iint r'\,d\omega\Sigma\frac{(\kappa - 3)\,\mathrm{L}(\kappa - f - 4)}{\mathrm{L}(f)\,\mathrm{L}(\kappa - 2f - 3)}\left(\frac{2\mathrm{F}}{\mathrm{E}}\right)^{\kappa - 3 - 2f}\left(\frac{1}{\mathrm{ED}^2}\right)^f,$$

where Σ implies summation from $f = 0$ to $\frac{1}{2}(\kappa - 3)$.

In these integrations l, m, n are the direction cosines of the moving radius vector r', and x, y, z are the coordinates of the fixed attracted point P. Let $\lambda = \alpha x$, $\mu = \beta y$, $\nu = \gamma z$, for the sake of brevity, then λ, μ, ν are constant during the integration.

Let

$$\mathrm{G} = \iint r'\,d\omega\,(l\lambda + m\mu + n\nu)^{2s}$$

where $2s = \kappa - 3$. As already explained, the integration extends over all positions of the moving radius vector round P.

Remembering that $\mathrm{F} = l\lambda + m\mu + n\nu$, we see that each term of the series for V may be derived from the preceding by a uniform process. Let

$$\Delta = \alpha\,d^2/d\lambda^2 + \beta\,d^2/d\mu^2 + \gamma\,d^2/d\nu^2.$$

After some slight numerical reductions we find

$$\mathrm{V} = \frac{\mu}{\kappa - 1}\frac{1}{2^{\kappa - 2}}\left[\left(\frac{4}{\mathrm{E}}\right)^{\kappa - 3}\mathrm{G} + \left(\frac{4}{\mathrm{E}}\right)^{\kappa - 4}\Delta\mathrm{G}\frac{1}{\kappa - 4} + \left(\frac{4}{\mathrm{E}}\right)^{\kappa - 5}\frac{\Delta^2\mathrm{G}}{\mathrm{L}(2)}\frac{1}{(\kappa - 4)(\kappa - 5)} + \dots\right.$$
$$\left. + \left(\frac{4}{\mathrm{E}}\right)^{\kappa - 3 - f}\frac{\Delta^f\mathrm{G}}{\mathrm{L}(f)}\cdot\frac{\mathrm{L}(\kappa - f - 4)}{\mathrm{L}(\kappa - 4)} + \dots\right]$$

where f varies from 0 to $\frac{1}{2}(\kappa - 3)$, so that the series for the potential has $\frac{1}{2}(\kappa - 1)$ terms, κ being the inverse index of the law of force.

When G has been expressed as a function of λ, μ, ν, and the axes of the homœoid whose attraction is required, this formula will give the potential at all internal points.

53. The value of G has been given by the double integral

$$\mathrm{G} = \iint r'\,d\omega\,(l\lambda + m\mu + n\nu)^{2s}$$

where $2s = \kappa - 3$. We shall now show that it can be reduced to a single integral.

Let the moving radius vector r' be called OR. The direction cosines of OR, when referred to the axes of the attracting homœoid, are l, m, n; let its direction cosines referred to the axes of the auxiliary ellipsoid be l', m', n'. Let the fixed straight line whose direction cosines are proportional to λ, μ, ν, when referred to the axes of the homœoid be called OL. Let its direction cosines, when referred to the axes of the auxiliary ellipsoid, be λ', μ', ν'. Then

$$G = \iint r'\, d\omega\, (\cos \mathrm{ROL})^{2s} (\lambda^2 + \mu^2 + \nu^2)^s.$$

Proceeding as in the lemma (Art. 49) we have

$$\iint r'\, d\omega\, (\cos \mathrm{ROL})^{2s} = \iiint dr'\, d\omega\, (\cos \mathrm{ROL})^{2s},$$

the integration extending throughout the volume of the auxiliary ellipsoid. If x', y', z' be the coordinates of a point on OR referred to the axes of the auxiliary ellipsoid, this is the same as

$$\iiint \frac{dx'\, dy'\, dz'}{r'^2} \left(\frac{\lambda' x' + \mu' y' + \nu' z'}{r'}\right)^{2s}.$$

Putting $x' = a'\xi$, $y' = b'\eta$, $z' = c'\zeta$, where a', b', c' are the semi-axes of the auxiliary ellipsoid, and transforming back to polar coordinates, we have

$$a'b'c' \iiint \frac{r'^2\, dr'\, d\omega\, (\lambda' a' \xi + \mu' b' \eta + \nu' c' \zeta)^{2s}}{(a'^2 \xi^2 + b'^2 \eta^2 + c'^2 \zeta^2)^{s+1}},$$

the integration extending throughout the volume of a sphere of radius unity. Putting $\xi = r'l'$, $\eta = r'm'$, $\zeta = r'n'$, and integrating from $r' = 0$ to 1, we find

$$\iint r'\, d\omega\, (\cos \mathrm{ROL})^{2s} = a'b'c' \iint \frac{(a'\lambda' l' + b'\mu' m' + c'\nu' n')^{2s}\, d\omega}{(a'^2 l'^2 + b'^2 m'^2 + c'^2 n'^2)^{s+1}}.$$

Referring to the lemma proved in Art. 10, and putting $t = 0$, we find that the right-hand side of this equation is

$$\frac{\mathrm{L}(2s)}{\mathrm{L}(s)\,\mathrm{L}(s)} \frac{2\pi}{2^{2s}} \int_0^\infty v^{-\frac{1}{2}}\, dv \left\{\frac{a'^2 \lambda'^2}{a'^2 + v} + \frac{b'^2 \mu'^2}{b'^2 + v} + \frac{c'^2 \nu'^2}{c'^2 + v}\right\}^s \frac{a'b'c'}{(a'^2 + v)^{\frac{1}{2}} (b'^2 + v)^{\frac{1}{2}} (c'^2 + v)^{\frac{1}{2}}}.$$

Returning to the notation that α', β', γ' (Art. 49) are the squared reciprocals of the axes of the auxiliary ellipsoid, and writing $v = 1/u$, we have

$$\iint r'\, d\omega\, (\cos \mathrm{ROL})^{2s} = \frac{\mathrm{L}(2s)}{\mathrm{L}(s)\,\mathrm{L}(s)} \frac{2\pi}{2^{2s}} \int_0^\infty \mathrm{M}^s \frac{u^s\, du}{\mathrm{Q}'},$$

where

$$Q'^2 = (\alpha' + u)(\beta' + u)(\gamma' + u),$$

$$M = \frac{\lambda'^2}{\alpha' + u} + \frac{\mu'^2}{\beta' + u} + \frac{\nu'^2}{\gamma' + u}.$$

The separate values of the axes $\alpha'\beta'\gamma'$ of the auxiliary ellipsoid may be found by the solution of a cubic, and if this were sufficient, G is here expressed as a single integral. But the product Q′ has been found in Art. 50 as a determinant without requiring the solution of the cubic. We shall now seek for an expression for M of the same kind.

54. Since α', β', γ' are the squared reciprocals of the axes of the auxiliary ellipsoid,

$$S = (\lambda X + \mu Y + \nu Z)^2 + (\alpha X^2 + \beta Y^2 + \gamma Z^2) E = 1 \quad . \quad . \quad . \quad . \quad \text{(I.)},$$

it follows that $\alpha' + u$, $\beta' + u$, $\gamma' + u$ are the squared reciprocals of the axes of

$$S + u(X^2 + Y^2 + Z^2) = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(II.)}.$$

Since these quadrics are concyclic, they are also co-axial, and therefore λ', μ', ν' are the direction cosines of OL referred to the axes of the second quadric.

It immediately follows that M is the sum of the squares of the projections of the axes on the straight line OL and is, therefore, equal to the sum of the squares of the projections of any three conjugate diameters on the same straight line.

Let OP_1, OP_2 be any two conjugate semi-diameters of the section $\lambda x + \mu y + \nu z = 0$, and let OP_3 be the third conjugate for the ellipsoid II. Then since OL is perpendicular to the section containing OP_1, OP_2, the sum of the squares of the projections on OL is

$$M = OP_3^2 (\cos P_3OL)^2.$$

Let λ'', μ'', ν'' be the direction cosines of OP_3 referred to the axes of the homœoid. Then by equation (II.)

$$\frac{(\alpha E + u)\lambda''}{\lambda} = \frac{(\beta E + u)\mu''}{\mu} = \frac{(\gamma E + u)\nu''}{\nu} = \rho \text{ (say)},$$

$$\cos P_3OL = \frac{\lambda\lambda'' + \mu\mu'' + \nu\nu''}{(\lambda^2 + \mu^2 + \nu^2)^{\frac{1}{2}}},$$

$$\frac{1}{OP_3^2} = (\lambda\lambda'' + \mu\mu'' + \nu\nu'')(\lambda\lambda'' + \mu\mu'' + \nu\nu'' + \rho).$$

Substituting these values of OP_3 and $\cos P_3OL$, we find, after clearing of fractions,

$$M = \frac{N}{(u + \alpha E)(u + \beta E)(u + \gamma E) + N} \frac{1}{\lambda^2 + \mu^2 + \nu^2}$$

where

$$N = \lambda^2 (u + \beta E)(u + \gamma E) + \mu^2 (u + \alpha E)(u + \beta E) + \nu^2 (u + \alpha E)(u + \beta E).$$

If M were written in partial fractions, it must be the same as

$$M = \frac{\lambda'^2}{u + \alpha'} + \frac{\mu'^2}{u + \beta'} + \frac{\nu'^2}{u + \gamma'},$$

as found above. The factors of the denominator must therefore be

$$Q'^2 = (u + \alpha')(u + \beta')(u + \gamma').$$

This is obviously correct since the denominator is the expansion of the determinant

$$Q'^2 = \begin{vmatrix} u + \alpha E + \lambda^2 & \lambda\mu & \lambda\nu \\ \lambda\mu & u + \beta E + \mu^2 & \mu\nu \\ \lambda\nu & \mu\nu & u + \gamma E + \nu^2 \end{vmatrix},$$

which agrees with the determinant already found for Q' in Art. 50, when we substitute $\lambda = \alpha x$, $\mu = \beta y$, $\nu = \gamma z$.

Finally, substituting in the value of G, we have

$$G = \frac{L(2s)}{L(s)\,L(s)} \cdot \frac{2\pi}{2^{2s}} \int_0^\infty \left(\frac{N}{Q'^2}\right)^s \frac{u^s\,du}{Q'}$$

where $2s = \kappa - 3$ and N, Q' have the values given just above. In this way G has been found by a single definite integral in terms of the axes of the attracting homœoid and the coordinates of the attracted point.

55. The potentials of elliptic rings and discs follow from those obtained for an ellipsoidal shell or solid. The transformation may be effected in either of two ways. By making the axis of c equal to zero, the homœoid becomes a disc and the potential at any external or internal point may be deduced from the formulæ of Art. 12. In this case the surface density at any point ξ, η of the disc is proportional to $\rho\,(1 - \xi^2/a^2 - \eta^2/b^2)^{-\frac{1}{2}}$ where ρ is the density of the homœoid at the point ξ, η, ζ. In this transformation the law of force remains unaltered so that if the formulæ of Art. 12 are used, the index must be an even number.

Another method is to make the axis of c infinite, so that the homœoid becomes a cylinder. Since the potential of an infinite rod, of line density $m\,ds$, at a distance r is $\frac{1}{\kappa - 1}\,\frac{\Gamma\frac{1}{2}(\kappa - 2)\,.\,\Gamma(\frac{1}{2})}{\Gamma\frac{1}{2}(\kappa - 1)}\,\frac{1}{r^{\kappa-2}}$, we see that the potential of the homœoid, when the index of the law of force is κ, becomes proportional to that of an elliptic ring with a law of force whose index is $\kappa - 1$, provided the attracted point lies in the plane of the ring. In this way the potential of a thin elliptic ring, whose line density at the point ξ, η, is proportional to $pf(\xi/a, \eta/b)$, where p is the perpendicular from the centre on the tangent, may be deduced from the formulæ of Art. 12, when the law of force varies as an odd power of the distance.

As these results may be obtained by simple substitution, we shall for the present pass them over and consider a theorem which is derived from neither of these methods.

56. When the force varies as the inverse cube of the distance, it is known that the level surfaces of a homogeneous elliptic disc are confocal quadrics. The potential at any point may then be found in elliptic coordinates by considering the potential at some convenient point of each confocal (see a paper by Mr. WEBB, 'Quarterly Journal,' vol. 14, 1877).

We shall now prove that *if the law of force is the inverse* κ^{th} *power of the distance, and the surface density at* ξ, η *is* $\sigma = (1 - \xi^2/a^2 - \eta^2/b^2)^{\frac{1}{2}(\kappa-3)}$, *the level surfaces are confocal quadrics of the elliptic disc.*

Let P be any point on a confocal, and let the normal at P meet the plane of the disc in G, then PG is the axis of the cone enveloping the disc and having its vertex at P. Draw two consecutive chords through G; let one of these be SGS′ and let $d\theta$ be the angle between them. Then PG makes equal angles with the opposite generators PS, PS′ of the cone.

Take two elements of area at R, R′ on opposite sides of G, such that PG bisects the angle RPR′ as well as the angle SPS′, let GR $= r$, GR′ $= r'$. These elements attract P equally if the surface densities σ, σ' at R, R′ are such that $\dfrac{\sigma r d\theta\, dr}{r^\kappa} = \dfrac{\sigma' r' d\theta\, dr'}{r'^\kappa}$. Supposing this to be true, the resultant attraction of the two elements, and therefore that of the whole disc, will act along the normal PG.

Because PG bisects both the angles SPS′, RPR′, we have by elementary trigonometry

$$\frac{1}{r} - \frac{1}{r'} = \frac{1}{\text{GS}} - \frac{1}{\text{GS}'}$$

$$\frac{\text{SR}\,.\,\text{RS}'}{\text{PR}^2} = \frac{\sin \text{SPR}\,.\,\sin \text{RPS}'}{\sin \text{S}\,.\,\sin \text{S}'} = \frac{\sin \text{S}'\text{PR}'\,.\,\sin \text{R}'\text{PS}}{\sin \text{S}\,.\,\sin \text{S}'} = \frac{\text{SR}'\,.\,\text{R}'\text{S}'}{\text{PR}'^2}.$$

The first of these results shows that $dr/r^2 = dr'/r'^2$. Since PR/GR = PR′/GR′, the second shows that

$$\frac{\sigma}{\sigma'} = \frac{r^{\kappa-3}}{r'^{\kappa-3}} = \left(\frac{\text{SR}\,.\,\text{RS}'}{\text{SR}'\,.\,\text{R}'\text{S}'}\right)^{\frac{1}{2}(\kappa-3)}.$$

Produce CR, CR′ to cut the conic in T, T′. If D be the semi-diameter parallel to the chord SGS′, we have by a well-known theorem in conics

$$\frac{\mathrm{SR}\,.\,\mathrm{RS}'}{\mathrm{D}^2} = 1 - \frac{\mathrm{CR}^2}{\mathrm{CT}^2} = 1 - \frac{\xi^2}{a^2} - \frac{\eta^2}{b^2}$$

where ξ, η are the coordinates of R. There is a corresponding equation for the product $\mathrm{SR}'\,.\,\mathrm{R}'\mathrm{S}'$. The necessary relation between σ, σ', is therefore satisfied by the given law of density. This proof does not require κ to be an integer.

57. To find the potential of the disc, we place the attracted point P on the axis of the disc. If h be its distance from the centre C, the potential at P is

$$\mathrm{V}_\kappa = \frac{1}{\kappa - 1} \iint \frac{d\xi\, d\eta\, (1 - \xi^2/a^2 - \eta^2/b^2)^{\frac{1}{2}(\kappa-3)}}{(\xi^2 + \eta^2 + h^2)^{\frac{1}{2}(\kappa-1)}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

writing $\xi = ax$, $\eta = by$ thus becomes

$$\mathrm{V}_\kappa = \frac{ab}{\kappa - 1} \iint \frac{r d\theta\, dr\, (1 - r^2)^{\frac{1}{2}(\kappa-3)}}{\{(a^2 \cos^2\theta + b^2 \sin^2\theta)\, r^2 + h^2\}^{\frac{1}{2}(\kappa-1)}} \quad . \quad . \quad . \quad . \quad (2),$$

the limits being $r = 0$ to 1, $\theta = 0$ to 2π. This may also be written in the form

$$\mathrm{V}_\kappa = \frac{ab}{\kappa - 1} \iint \frac{r d\theta\, dr\, (1 - r^2)^{\frac{1}{2}(\kappa-3)}}{\{(a'^2 \cos^2\theta + b'^2 \sin^2\theta)\, r^2 + h^2 (1 - r^2)\}^{\frac{1}{2}(\kappa-1)}} \quad . \quad . \quad . \quad (3),$$

where $a'^2 = a^2 + h^2$, $b'^2 = b^2 + h^2$, so that a', b', h are the axes of the confocal through the attracted point. When $\kappa = 3$ the integral is easily found; we have

$$\mathrm{V}_3 = \pi \log \frac{a'b + ab'}{h\,(a + b)}\,.$$

We see from the third form above that

$$\frac{\mathrm{V}_\kappa}{ab} = \frac{1}{\mathrm{L}\,\frac{1}{2}(\kappa - 1)} \cdot (-1)^{\frac{1}{2}(\kappa-3)} \left(\frac{d}{dh^2}\right)^{\frac{1}{2}(\kappa-3)} \frac{\mathrm{V}_3}{ab},$$

the differentiation being effected on the supposition that a', b' are constant.

When the disc is circular, the first form for V_κ becomes

$$\mathrm{V}_\kappa = \frac{1}{\kappa - 1} \frac{1}{a^{\kappa-3}} \iint \frac{r d\theta\, dr\, (a^2 - r^2)^{\frac{1}{2}(\kappa-3)}}{(r^2 + h^2)^{\frac{1}{2}(\kappa-1)}}\,.$$

Putting $a^2 - r^2 = (a^2 + h^2) \sin^2\phi$, this immediately reduces to

$$\mathrm{V}_\kappa = \frac{2\pi}{\kappa - 1} \frac{1}{a^{\kappa-3}} \int_0^\gamma (\tan\phi)^{\kappa-2}\, d\phi,$$

where γ is the angle subtended by any radius of the disc at the intersection of the confocal with the axis of the disc. The integration can be easily performed by a formula of reduction.

58. Conversely, we may inquire whether this is the only law of density which makes the level surfaces of the disc to be confocals. Taking the law of force to be the inverse κ^{th} power of the distance where $\kappa > 3$, let P be any point on the confocal immediately adjacent to the disc. The potential at P is due to the matter of the disc in its immediate neighbourhood. Let σ be the surface density at this point, z the infinitely small distance of P from the disc. The potential at P of an infinite plane of this surface density is

$$\int \frac{2\pi r\, dr}{(\kappa - 1)\, r^{\kappa-1}}\, \sigma = \frac{2\pi\sigma}{(\kappa - 1)(\kappa - 3)}\, \frac{1}{z^{\kappa-3}}.$$

Hence the potential cannot be constant at all points of this confocal unless σ varies as $z^{\kappa-3}$, *i.e.*, σ varies as $(1 - \xi^2/a^2 - \eta^2/b^2)^{\frac{1}{2}(\kappa-3)}$.

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